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A Mathematical Outline of the Keynesian Macroeconomic Model of the Economy

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# A Mathematical Outline of the Keynesian Macroeconomic Model of the Economy

# Ian St John

#### Abstract

The purpose of this paper is to outline the essential equations of the Keynesian model of the economy. It begins with the simplest Keynesian model, with no government sector or external trade. We then incorporate a government sector, and then overseas trade. Although the addition of new forms of spending entering the circular flow of income, and new forms of leakage from the circular flow, affect the equilibrium level of National Income, the basic proposition remains: namely, that the equilibrium level of National Income depends upon the level of autonomous spending and the value of the Multiplier.<sup>1</sup>

National Income Identities in a Simple Keynesian Model with No Government or Overseas Trade

In a simple Keynesian macroeconomic model, the value of total money spending in the economy is always equal to the money value of total income. This is because every act of spending money generates income for the recipient of the money. And

<sup>&</sup>lt;sup>1</sup> This paper merely summarises well established results. In writing it I have consulted the following works: R. Lipsey, *An Introduction to Positive Economics* (Weidenfeld and Nicholson, London, 1966); I. Jacques, *Mathematics for Economics and Business* (Pearson, Harlow, Tenth Edition, 2023); M. Wisniewski, *Introductory Mathematical Methods in Economics* (McGraw-Hill, London, 1991); M. Wisniewski, *Mathematics for Economics* (Palgrave Macmillan, Basingstoke, 2013); and T. Bradley, *Essential Mathematics for Economics* (John Wiley, West Sussex, 2014).

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the money value of total income always equals the money value of total output, for it is the value of the output of the economy which is paid out as incomes to those involved in its production, whether as wages to workers, interest to financiers, rents to property owners, or profits to entrepreneurs. Thus:

$$Y \equiv E$$

(1)

Where:

Y = the total nominal value of the economic output of an economy produced during a period of time (usually a year). We assume that the money generated in producing output is all distributed as Incomes (wages, rents, profits etc) to those involved in producing it. So Y is the money value of the Output or Income of the economy.
 E = the total nominal value of Expenditure in the economy

This is an accounting identity, which means that the total value of Expenditure always equals the value of Output by definition, and all Output becomes Income. For example, if Expenditure is less than the value of Output, then firms will accumulate stocks and an addition to stocks is a form of Investment spending, so total spending will automatically be equal to the value of output.

 $E \equiv C + I \tag{2}$ 

C = Consumption spending by households

I = Investment spending by firms

In a simple Keynesian model there are two types of spending – Consumption spending by households and Investment spending by firms.

$$Y \equiv C + S \tag{3}$$

S = Saving by households.

Thus, all Income received by households is Consumed or Saved.

Since  $Y \equiv E$  then:

$C + S \equiv C + I$	(4)
$S \equiv I$	(5)

This means that in an economy, *actual* Saving always equals *actual* Investment.

# **Equilibrium Conditions**

Equilibrium National Income (Ye) is that level of National Income or Output that has no tendency to change. It exists when Planned Income (Output) equals Planned

Expenditure – when the value of output firms wish to produce equals the value of goods households and firms wish to buy.

 $Yp = Ep \tag{6}$ 

Here the subscripts p indicate 'planned'.

Since:

 $Yp = Cp + Sp \tag{7}$ 

 $Ep = Cp + Ip \tag{8}$ 

Then:

 $Cp + Sp = Cp + Ip \tag{9}$ 

Subtracting Cp from both sides:

 $Sp = Ip \tag{10}$ 

Thus, equilibrium National Income exists when Planned Saving = Planned Investment.

In the Keynesian model it is assumed that firms produce output in response to demand. Thus:

Ye = f(Ep)

Planned Expenditure causes Planned Income to adjust until the two are equal. If Ep > Yp, then Yp increases until Yp = Ye = Ep. And if Ep < Yp, then Yp contracts until Yp = Ye = Ep.

This raises the question: what determines Planned Expenditure?

# **Behavioural Functions**

#### Investment

Planned Investment is determined by Investment decisions of firms. In a simple Keynesian model Investment is assumed to be fixed and *autonomous*, which it to say Investment is determined outside the model and is not a function of Income. I.e.

Ip = Io	(11)
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# Consumption

Keynes put forward idea of a Consumption Function, which says that Planned Consumption by households is a function of Income:

$$Cp = a + bY \tag{12}$$

Cp = planned Consumption by households.

- a = autonomous consumption spending by households which does not depend upon income. It is the amount of consumption spending households do when Income is zero. For example, a household with no income will spend out of savings. It is assumed to be fixed.
- bY = consumption that is determined by income. b is the share of an increase in income that is consumed. It is the *Marginal Propensity to Consume* (mpc).

Differentiating Consumption by Income we get:

$$\frac{dC}{dY} = \frac{d}{dY}(a+bY) = b$$
(13)

where b = Marginal Propensity to Consume (mpc)

## Saving

All Income received by households is either Consumed or Saved:

 $\mathbf{Y} = \mathbf{C} + \mathbf{S}$ 

Hence when determining their Planned Consumption, households are automatically determining their Planned Saving:

$$Sp = Y - Cp \tag{14}$$

Sp is Planned Saving and equal to Income minus Planned Consumption spending.

$$Sp = Y - (a + bY)$$
  
 $Sp = Y - a - bY$   
 $Sp = -a + Y(1 - b)$  (15)

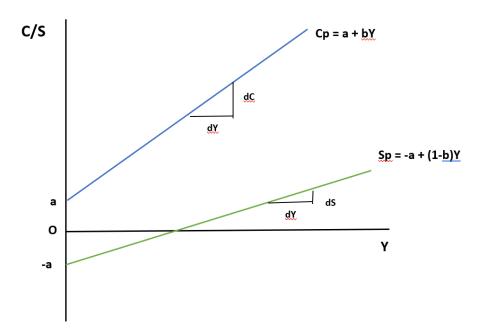
Note that autonomous Consumption a has become -a, representing dissaving.

$$\frac{\mathrm{dS}}{\mathrm{dY}} = 1 - \mathrm{b} \tag{16}$$

1 – b is the Marginal Propensity to Save (mps). I.e.

$$MPS = 1 - MPC$$

These relationships are illustrated in Figure 1.





As can be seen, as Income increases so Consumption increases at a constant rate dC/dY, which is the mpc, while Saving increases at the rate dS/dY, which is the mps.

For example: if Cp = 10 + 0.6Y

Then: Sp = -10 + (1 - 0.6)Y = -10 + 0.4Y

## **Determining Equilibrium National Income**

Equilibrium National Income exists when the planned spending by households and firms equals the planned output by firms. That is:

Yp = Ep(6) Yp = Cp + Ip(7) Since Cp = a + bY and I = Io, then: Y = a + bY + Io Y - bY = a + Io Y(1 - b) = a + Io $Ye = \frac{a + Io}{(1 - b)}$ (17)

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$$Ye = \frac{1}{1-b} (a + Io)$$

$$Ye = \frac{1}{1-mpc} (a + Io)$$
(18)
$$Ye = \frac{1}{mps} (a + Io)$$
(19)

Here the ratio 1/1-mpc or 1/mps is designated the *multiplier* (K). Hence:

$$Ye = K (a + Io)$$

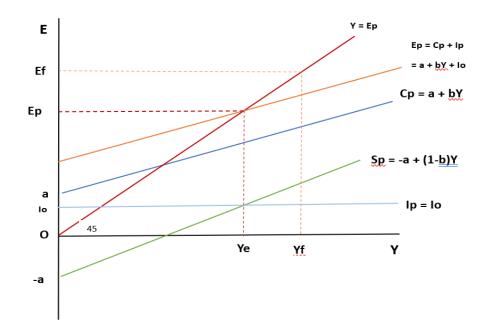
That is:

Ye = K (Autonomous spending)

So the equilibrium level of National Income Ye depends upon:

- (1) Planned Investment Io
- (2) Autonomous Consumption a
- (3) mpc (or mps) and hence the value of the multiplier (1/1-mpc).

These points are illustrated in Figure 2 below.





This diagram shows the determination of Equilibrium National Income in the Keynesian system. The diagonal 45 degree line shows all combinations where E = Y. The equilibrium level of National Income Ye depends on the level of Planned Expenditure Ep. Planned Expenditure is made up of Planned Consumption and Planned Investment. It intercepts the y-axis at a + Io. Its slope is determined by the mpc (=b). Here equilibrium Income as determined by Ep is Ye. Planned Output = Planned Expenditure. At this point it can also be seen that Planned Investment = Planned Saving. However, equilibrium National Income is *not* the same as Full Employment National Income, where all available resources are employed. They may coincide but probably will not. For example, in **Figure 2**, Full Employment Income is Yf, corresponding to Full Employment Expenditure Ef. So in this case equilibrium Income Ye < Yf and there are unemployed resources – a *deflationary gap*. If Ye > Yf there will be an *inflationary gap*.

#### **Comparative Static Results**

What happens to equilibrium National Income if a or Ip change?

$$\frac{\partial Y}{\partial a} = \frac{\partial}{\partial a} \left( \frac{a+lp}{1-b} \right) = \frac{1}{1-b}$$
(20)  
$$\frac{\partial Y}{\partial lp} = \frac{\partial}{\partial lp} \left( \frac{a+lp}{1-b} \right) = \frac{1}{1-b}$$
(21)

Accounting for changes in National Income in terms of a differential equation we get:

$$dY = \frac{\partial Y}{\partial a} da + \frac{\partial Y}{\partial Ip} dIp$$
$$dY = \left(\frac{1}{1-b}\right) da + \left(\frac{1}{1-b}\right) dIp$$
(22)

The ratio of dY/da and dY/dIp is the *multiplier*.

$$\frac{dY}{da} = \left(\frac{1}{1-b}\right) = \frac{1}{1-mpc} = \text{the multiplier} = K$$
(23)  
$$\frac{dY}{dIp} = \left(\frac{1}{1-b}\right) = \frac{1}{1-mpc} = \text{the multiplier} = K$$
(24)

#### A Numerical Example

Suppose:

Cp = 200 + 0.5Y

Ip = 800

Then Equilibrium National Income is:

Y = EpY = Cp + IpY = 200 + 0.5Y + 800Y = 1000 + 0.5YY - 0.5Y = 1000Y(1 - 0.5) = 1000 $Y = \frac{1000}{1 - 0.5} = \frac{1000}{0.5} = 2000$ So Ye = 2000. We can confirm that at Ye = 2000 then Sp = Ip. Given that Cp = 200 + 0.5Y, then: Sp = Y - CpSp = Y - (200 + 0.5Y)Sp = -200 + (1 - 0.5)YSp = -200 + 0.5YSp = -200 + 0.5(2000)Sp = -200 + 1000Sp = 800 = Ip

Thus, as expected, in equilibrium Planned Investment = Planned Saving.

Suppose Investment increases from 800 to 1000. Then:

Y = Cp + Ip Y = 200 + 0.5Y + 1000Y = 1200 + 0.5Y

Y - 0.5Y = 1200 Y(1 - 0.5) = 1200 Ye =  $\frac{1200}{0.5}$  = 2400 Checking that Sp = Ip Sp = -200 + (1 - 0.5)Y Sp = -200 + (0.5)2,400 Sp = -200 + 1,200 Sp = 1000 = Ip In this case we see that:  $\Delta I = 200$   $\Delta Y = 400$ So:  $\frac{\Delta Y}{\Delta I} = \frac{400}{200} = 2 = K$ 

An increase in Investment of 200 has led to an increase in National Income of 400. This is ratio of 400/200 = 2 is the multiplier, which is also equal to:

 $\frac{1}{1 - mpc} = \frac{1}{1 - 0.5} = \frac{1}{0.5} = 2$ 

# A Keynesian Model with Government

Thus far we have considered the simplest Keynesian National Income model, with no government or foreign trade. Let us now introduce a government sector into our model.

We assume that government spending G is autonomous – in other words, that it is not determined within our model but fixed at some level Go by the government. In which case, total planned Expenditure is:

Ep = Cp + Ip + Gp(25)

Assume that Ip and Gp are fixed, so:

Ep = Cp + Io + Go(26)

Government spending is raised from Taxation (T). Assume this is levied at a fixed rate t on Income. So:

$$T = tY$$
(27)

Where t is the *Marginal Rate of Taxation* and  $0 \le t < 1$ .

Consumption now takes place out of Income after tax has been taken; this is

Disposable Income, Yd.

Yd = Y - T

So:

Yd = Y - tY

$$Yd = (1 - t)Y$$
 (28)

Thus the Consumption Function with tax is:

$$C = a + b(Yd)$$
  
 $C = a + b[(1 - t)Y]$  (29)

Equilibrium National Income now becomes:

$$Y = Cp + Io + Go$$

$$Y = a + b[(1 - t)Y] + Io + Go$$

$$Y - b[(1 - t)Y] = a + Io + Go$$

$$Y[1 - b(1 - t)] = a + Io + GO$$

$$Ye = \frac{a + Io + Go}{1 - b(1 - t)}$$

$$Ye = \frac{1}{1 - b(1 - t)} (a + Io + Go)$$
(31)

Thus in a Keynesian closed economy with Government, equilibrium National Income depends on:

- 1. The levels of autonomous spending a, Io, and Go.
- 2. The value of b or the mpc the higher is b, the higher is Y other things being equal.
- 3. The value of t. The level of National Income is *inversely related* to the value of t, which is the share of Income taken as Tax.

#### **Comparative Static Results**

Again we can take the first partial differential of Y with respect to a change in any of the autonomous forms of spending. In the case of a change in Government spending G we have:

$$\frac{\partial Y}{\partial G} = \frac{\partial}{\partial G} \left( \frac{a + Io + G}{1 - b(1 - t)} \right) = \frac{1}{1 - b(1 - t)}$$
(32)

So the change in National Income from a given change in G is:

$$\Delta Y = \frac{1}{1 - b(1 - t)} \Delta G \tag{33}$$

Thus:

 $\frac{\Delta Y}{\Delta G} = \frac{1}{1 - b(1 - t)}$  = the value of the multiplier = K. (34)

The higher is t the lower is the value of the multiplier. If t = 0 then the multiplier is 1 - b(1 - 0) = 1 - b, which is the value of the multiplier without taxation. If, for example, t = 0.1 then the multiplier is 1 - b(1 - 0.1) = 1 - 0.9b. Since we are subtracting a lower number from 1, the number we get is higher than before, so the multiplier which is 1/1 - b(1 - t) will decline. To illustrate: if b = 0.5 and t = 0, then the multiplier is 1/1 - 0.5 = 1/0.5 = 2. But if t = 0.4 then the multiplier is: 1/1 - 0.5(1 - 0.4) = 1/1 - 0.5(0.6) = 1/1 - 0.3 = 1/0.7 = 1.4.

This shows that the effect of introducing a positive tax rate is to lower the value of the multiplier, and the higher the tax rate t the lower the multiplier. If the tax rate were 1 and all income were taxed then the multiplier would be 1/1 - b(0) = 1/1 = 1 and there would be no multiplier effect at all.

By taking the first partial derivative of Ye with respect to a change in the tax rate t, we can prove that the equilibrium level of national income will rise when tax rates fall, and fall when tax rates rise. Thus, if:

$$Ye = \frac{a + Io + Go}{1 - b(1 - t)}$$

then (using the quotient rule):

$$\frac{\partial Ye}{\partial t} = \frac{1 - b(1 - t)(0) - (a + Io + Go)(b)}{1 - b(1 - t)^2} = \frac{-(a + Io + Go)b}{1 - b(1 - t)^2}$$
(35)

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Since the value of the parameters here mean that the denominator must be positive and the numerator is negative, then  $\partial Ye/\partial t < 0$ , so that Ye and t are inversely related: a rise in the rate of tax will cause national income to fall, and a fall in t will cause Ye to rise.

# A Keynesian National Income Model with Government and Foreign

# Trade

To complete our Keynesian model, let us add a foreign trade sector. This brings

two changes to the model:

- 1. We now have a new form of autonomous spending, in the form of Export Demand, which is a new injection of spending into the economy, X.
- 2. We have a new leakage of spending from the economy in the form of spending on Imports, which we designate M.

Imports M are a dependent variable, determined within the system as a function of Income. We assume that a portion m of disposable income after tax Yd is spent on imported goods. We call m the *Marginal Propensity to Import*. Thus:

(36)

M = m(Yd)

The net addition to total expenditure E from the inclusion of foreign trade is thus:

Exports minus Imports

X – M

It follows that equilibrium National Income exists when:

Ye = C + I + G + (X - M)(37)

Where C, I, G, X, and M are all planned values. The reason we subtract M in this equation is because some of Consumption spending C will be on Imports (M), so we must subtract M from total spending on domestic output.

This can be written more fully as:

Ye = a + b(Yd) + I + G + (X - mYd) (38)

We have already seen that:

Yd = Y - T

Yd = Y - tYYd = (1 - t)Y (28)

Inserting this expression for Yd into our equilibrium condition we have:

$$Y = a + b[(1 - t)Y] + I + G + X - m[(1 - t)Y]$$
(39)

Rearranging the Y terms together we have:

$$Y - b[(1 - t)Y] + m[(1 - t)Y] = I + G + X$$

$$Y[1 - b(1 - t) + m(1 - t) = I + G + X$$

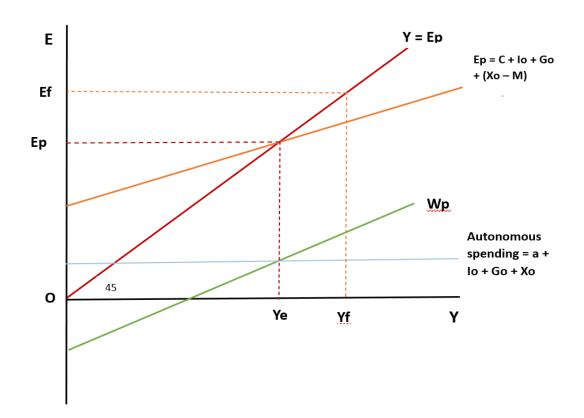
$$Y = \frac{I + G + X}{1 - b(1 - t) + m(1 - t)}$$

$$Y = \frac{I + G + X}{1 - b + bt + m - mt}$$

$$Y = \frac{I + G + X}{1 - (1 - t)(b - m)}$$
(40)

In this case, in terms of the numerator of this expression, the higher is I, G, and X, the higher is equilibrium National Income Ye. With respect to the denominator, the point is that the *lower* is the denominator the higher national income. So taking the first parentheses we see that the higher is t, the lower is 1 - t, and so the higher is 1 - (1 - t), so the *lower* is National Income. Equally, the higher is m (the Marginal Propensity to Import) relative to b (the Marginal Propensity to Consumer) the lower is b - m and so the higher is 1 - (1 - t)(b - m) and the *lower* is National Income. Put simply, National Income Y is *positively* related to b and *inversely* related to t and m. The higher is the rate of tax and the Marginal Propensity to Import, the lower is the equilibrium level of National Income Ye, which is what we would expect given that Consumption spending passes spending on within the economy whereas imports cause spending to leave the domestic economy.

Our Keynesian cross diagram showing the determination of equilibrium National Income can now be represented as follows.



# Figure 3. Determining Equilibrium National Income with Government and External Trade

This diagram is similar to that encountered in **Figure 2**. The difference is that total Expenditure now includes Government spending and Export demand in addition to Consumption spending. In this case total spending yields equilibrium Income of Ye. Wp represents the total withdrawals function, which is Saving plus Imports plus Tax revenue, all of which are rising functions of Income. At equilibrium income Ye, total Withdrawals are equal to planned Injections, which is planned autonomous spending.

The change in Y with respect to a change in Exports (X) alone is:

 $dY = \frac{\partial}{\partial X} \left( \frac{a + Io + G + X}{1 - (1 - t)(b - m)} \right) dX$  $dY = \frac{1}{1 - (1 - t)(b - m)} dX$  $\frac{dY}{dX} = \frac{1}{1 - (1 - t)(b - m)} = \text{the multiplier} = K$ (41)

Thus, the higher are t and m, the lower the value of the multiplier.

# A Numerical Example

We have seen that in an open economy with government:

$$Y = a + b[(1 - t)Y] + I + G + X - m[(1 - t)Y]$$
(38)

Suppose that:

a = 200

b = 0.5

t = 0.2

m = 0.1

I = 800

G = 200

X = 200

We can calculate National Income Y as follows:

$$Y = 200 + 0.5[(1 - 0.2)Y] + 800 + 200 + 200 - 0.1[(1 - 0.2)Y]$$
  

$$Y = 1400 + 0.5[(0.8)Y] - 0.1[(0.8)Y]$$
  

$$Y = 1400 + 0.4Y - 0.08Y$$
  

$$Y = 1400 + 0.32Y$$
  

$$Y - 0.32Y = 1400$$
  

$$Y(1 - 0.32) = 1400$$
  

$$Y(0.68) = 1400$$
  

$$Ye = \frac{1400}{0.68} = 2058.8$$

Thus equilibrium National Income is 2058.8

We can check that at this level of Income that planned withdrawals equal planned autonomous spending.

Planned autonomous spending = a + I + G + X = 200 + 800 + 200 + 200 = 1400

Planned withdrawals = S + T + M

$$S = 0.5[(1 - t)Y]$$
  
T = 0.2Y

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M = m[(1 - t)Y]Thus S = 0.5[(1 - 0.2)Y] = 0.5[(0.8)Y] = 0.4Y T = 0.2Y M = 0.1[(1 - 0.2)Y] = 0.1[(0.8)Y] = 0.08YSo with Y = 2058.8 we get: S = 0.4(2058.8) = 823.52 T = 0.2(2058.8) = 411.76 M = 0.08(2058.8) = 164.7

Thus total withdrawals equal:

823.52 + 411.76 + 164.7 = 1400

Which confirms that Planned Withdrawals = Autonomous Spending = 1400.

In this case the Multiplier (K) equals:

$$K = \frac{1}{1 - (1 - t)(b - m)}$$

$$K = \frac{1}{1 - (1 - 0.2)(0.5 - 0.1)} = \frac{1}{1 - (0.8)(0.4)} = \frac{1}{0.68} = 1.47$$

Thus the effect of introducing taxation and import spending as 'leakages' from the circular flow of income has been to reduce the value of the multiplier from 2 to 1.47. In this case, equilibrium National Income is also equal to:

Ye = K x (autonomous spending)

Ye = (1.47)(1400) = 2058

Which confirms our calculation above.

#### Conclusion

What our summary of the simple Keynesian National Income model shows is that:

 Equilibrium National Income or Output depends on the total level of Planned Expenditure in the economy. Firms produce output to meet demand. In the simplest version of the model, this Expenditure is Consumption spending and Investment spending only. As we elaborated our model we added two new forms of spending, namely Government spending and Export demand.

- 2. However, it is not all Expenditure which determines Income since some spending (namely part of Consumption spending) is a function of Income. The spending that *determines* equilibrium National Income is *autonomous* spending that is, spending that is not a function of income. This consists of: autonomous consumption spending (a), Investment spending (Io), and Export spending X.
- 3. Autonomous spending determines National Income *via* the Multiplier (K). That is:

Ye = K x (Autonomous Spending)

Where the value of the multiplier is determined by the Marginal Propensity to Consume (or Marginal Propensity to Save), the Marginal Propensity to Import, and the Marginal Rate of Tax.

- 4. As we expand the Keynesian model from a closed economy with no government, to a closed economy with government, and then to an open economy with government, the value of the multiplier steadily diminishes. The is because of any increase in income received by households, less is passed on as further increased spending as some leaks away as government tax revenue (the Marginal Rate of Taxation) and some leaks outside the economy as spending on Imports (the Marginal Propensity to Import).
- 5. Similarly, changes in equilibrium National Income are explained by changes in autonomous spending in conjunction with the multiplier. I.e.

 $\Delta$ Ye = K x (Change in Autonomous Spending)

6. Equilibrium National Income must be distinguished from the Full Employment level of National Income. An economy will tend to equilibrium National Income where total output equals total spending (Ye = Ep). Once it has arrived at equilibrium there will no tendency for National Income to change unless either autonomous spending changes or the value of the multiplier changes. But there is no reason to assume that equilibrium Income (Ye) will correspond with that level of National Income at which all resources (including labour) are employed. Hence it is possible for an economy to be in equilibrium *with* unemployment. This was the point Keynes wished to establish in the 1930s where he argued that there was no inherent tendency within a free market economy to ensure that there will be full employment of resources.