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Harrod's Model of Economic Growth

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Abstract

Since the classical economists in the nineteenth century, attempts to explain the growth of an economy had been lost sight of – attention switching, instead, to what determines equilibrium prices and quantities within an economy. Keynes's *General Theory of Employment, Interest and Money* (1936) shifted attention back to the total output of an economy, but again within an equilibrium, comparative-static, framework. Yet Keynes's conceptual tools – total saving, the consumption function, investment, and so forth – provided the elements out of which Roy Harrod was able to construct, in 1939, a model of economic growth within the Keynesian tradition. Harrod's model was one of the first to explicitly consider the economy, not in terms of static variables, but in terms of continually growing variables. His goal was to develop out of Keynes's work a *dynamic* model, framed in terms of rates of change. This paper explains each of Harrod's growth concepts: actual growth, warranted or equilibrium growth, and the natural rate of economic growth, in relation to his fundamental growth equation, which remains a staple of textbooks to this day. Equilibrium growth is shown to be precarious, resting on two 'knife-edge' conditions which, when ceasing to hold, can yield growth which is unstable and tending to fall below the level necessary for full employment. It concludes by noting some leading criticisms of Harrod's work.

Sir Roy Harrod's model of economic growth, first put forward in his 1939 article 'An Essay in Dynamic Theory', inaugurated modern macroeconomic growth theory.[1] Published only three years after Keynes' *General Theory of Employment, Interest and Money*, and corrected by Keynes himself in his capacity as editor of the *Economic Journal* (where it appeared), Harrod's model was an early product of the Keynesian Revolution. It sought to take the basic variables and relations of Keynesian macroeconomics (Consumption, Saving, Investment, National Income) and situate

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[1] R.F. Harrod, 'An Essay in Dynamic Theory', in R.F. Harrod, *Economic Essays* (Macmillan, London, 1952), pp. 254-277.

them in a *dynamic* context. Where Keynes had identified the static conditions for the equilibrium level of national income, Harrod wished to formulate the conditions for equilibrium in a dynamic economy, a dynamic economy being defined as one in which 'rates of output are changing.' Like Keynes, Harrod saw an economy as being in equilibrium when Planned Saving = Planned Investment. The difference was that, for Harrod, the equilibrium was not an equilibrium level of national income, but rather an *equilibrium rate of growth of national income*. This equilibrium growth rate, where Planned Saving = Planned Investment, he calls the *warranted rate of growth* or G_W. This warranted growth rate is to be distinguished from the *actual* growth rate, G_A, and the *natural* long-run growth rate G_N. An analysis in terms of growth rather than static equilibrium was a necessary corollary of Keynes's work for the simple reason that an economy which saves and invests will also grow, since investment means more capital and hence more output going forward. The question, then, is what are the conditions for equilibrium warranted growth to be achieved, and what are the consequences if these conditions are not fulfilled?

Harrod's Fundamental Equation for Actual Economic Growth

We first outline how Harrod's fundamental growth equation can be used to model actual economic growth, G_A. The actual growth rate is that rate of growth when actual saving equals actual investment. To simplify, Harrod assumed that there was no capital depreciation (so capital does not deteriorate over time, so absolute and net investment are the same) and that technical progress leaves the capital-output ratio unchanged. He also assumed that any part of national income or total output (Y) that is not consumed in a period of time is invested. I.e.

$Y \equiv C + I$

$Y - C \equiv I$

where Y is real national output or income, C is total consumption by households, and I is investment by firms. Since $Y - C \equiv S$, then it is *always* true that $I \equiv S$, which means that actual investment equals actual saving – a standard Keynesian assumption. With these assumptions, Harrod proceeds as follows.

(1) The actual rate of growth (G_A) of national income or total output (Y) is:

$$G_A = \frac{\Delta Y}{Y}$$

where Y is the level of total output at period t and ΔY is the increase in output between periods t and t+1, i.e. $\Delta Y = Y_{t+1} - Y_t$.

(2) Total saving (S) by a community during period t is a constant proportion (*s*) of the total income (Y) received during that period of time. That is:

S = sY

Saving, then, is a function of income, where is the marginal propensity to save. That share of an increase income that is saved had been called by Keynes the marginal propensity to save (MPS). Since saving is a function of income only, then the MPS is the same as the average propensity to save (APS). In other words:

S = sY

$$s = \frac{S}{Y} = MPS = APS$$

Harrod assumed that S is the amount that people *plan* to save from a given amount of income, and that planned saving equals actual saving.[1]

(3) All savings are invested and therefore actual investment equals actual (and planned) saving: I = S. Whatever output is not consumed is considered to be invested. That is:

Y – C = S = I

(4) Since there is no depreciation of capital assets, investment adds to the stock of capital in the economy, i.e. I = ΔK, where K is the capital stock.

(5) Because I = ΔK and S = sY, then the I = S condition can be written:

ΔK = sY

This, again, states that actual investment equals actual saving.

(6) Harrod defined the capital/output ratio (v) as the increase in capital (ΔK) associated with the increase in income (ΔY) in a given period. In terms of *actual growth*, v is the ratio of the *actual* increase in the capital stock to the *actual* increase in income. We designate this *actual* capital/output ratio v_A. That is:

$$v_A = \frac{\Delta K}{\Delta Y} = \frac{I}{\Delta Y}$$

(7) Since $v_A = \frac{\Delta K}{\Delta Y} = \frac{I}{\Delta Y}$ then ΔY = $\frac{\Delta K}{v_A}$

This means that a given increase in K divided by the capital/output ratio will yield a defined increase in output or income. It follows that ΔK = I = v_AΔY. This is a basic 'accelerator' model of investment, where investment is a function of changes in income or output.

(8) Given (7), the actual rate of growth ΔY/Y can be expressed as:

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[1] We are assuming s is fixed. In reality, says Harrod, it will vary with 'the size of income, the phase of the trade cycle, institutional changes, etc.' (Harrod, 'An Essay in Dynamic Theory', p. 257).

$$\frac{\Delta Y}{Y} = \frac{\frac{\Delta K}{v_A}}{Y} \qquad \text{or, since } \Delta K = I,$$

$$\frac{\Delta Y}{Y} = \frac{\frac{I}{v_A}}{Y}$$

(9) We have already seen (5) that $\Delta K = sY$, so:

$$\frac{\Delta Y}{Y} = \frac{\frac{sY}{v_A}}{Y}$$

It follows that:

$$\frac{\Delta Y}{Y} = \frac{sY}{v_A} \cdot \frac{1}{Y} = \frac{s}{v_A}$$

(10) Thus, we arrive at Harrod's 'Fundamental Growth Equation' for actual economic growth:

$$G_A = \frac{\Delta Y}{Y} = \frac{s}{v_A} = \frac{\text{MPS}}{\text{Actual capital/output ratio}}$$

(11) We know that $s = \frac{S}{Y}$ and $v_A = \frac{I}{\Delta Y}$, hence (10) can be re-written:

$$\frac{\Delta Y}{Y} = \frac{\frac{S}{Y}}{\frac{I}{\Delta Y}}$$

Multiplying both sides by $I/\Delta Y$ we get:

$$\frac{\Delta Y}{Y} \frac{I}{\Delta Y} = \frac{S}{Y}$$

$$\frac{I}{Y} = \frac{S}{Y}$$

$$I = S$$

Thus, for an economy growing at the actual rate G_A, actual investment equals actual saving, which is a familiar result in Keynesian economics.

(12) We can derive Harrod's growth equation more simply from the assumption that, in any economy at any given time, actual investment equals actual saving. Since $S = sY$ and $I = v_A \Delta Y$, then:

$$sY = v_A \Delta Y$$

in which case:

$$\frac{\Delta Y}{Y} = \frac{s}{v_A} = G_A$$

In this form, Harrod's equation is said to be a 'truism'. All we have said is that G_A *must* equal s/v_A. In that sense, we can write:

$G_A \equiv s/v_A$

This is an accounting identity rather than a theory. For example, if *s* (the MPS and APS) rises, then either the capital/output ratio V_A must rise, or G_A must rise, or a combination of both. And if the actual rate of growth increases, we know that either *s* has increased or the capital/output ratio has fallen. Most important: although *actual* investment equals *actual* saving, this does *not* mean that planned investment equals planned saving. While Harrod assumes that actual saving equals planned saving, actual investment is not assumed to be equal to planned investment. Actual investment depends on actual saving since any output not consumed is assumed to be invested: I = ΔK = sY or S, so if S rises then ΔK rises or *vice versa*. But actual investment does not necessarily equal planned investment (I = ΔK = sY = S, so if S rises then ΔK rises or *vice versa*). Hence, an increase in saving can simply result in an increase in the amount of Investment associated with a change in income (a rise in V_A). Put simply, for any observed rate of actual economic growth, G_A, while actual investment equals actual saving (I = S), we cannot say that planned Investment equals Planned Saving (Ip = Sp), and thus we cannot say that the actual rate of growth is the *equilibrium* rate of growth – or what Harrod called the *warranted* rate of growth G_W. It may be; but it is more likely not to be!

Harrod's Fundamental Equation for Equilibrium (Warranted) Economic Growth

The set of equations we used to derive the actual rate of growth can also be used to derive the equilibrium rate of growth – with one difference. This is that the capital/output ratio is now fixed. We call this fixed capital/output ratio the *required* capital/output ratio v_R. This required ratio is determined by technology. It is the increase in capital entrepreneurs *need and require* to increase output by a given amount.[1] It is determined by the marginal product of capital (the marginal efficiency of investment), which is in turn determined by prevailing technology. What it says is how much firms need to invest to increase output by one unit.[2] It is this that determines planned investment, as firms respond to increasing demand for output by increasing their capital stock according to the technically fixed capital/output ratio. Thus, we have the warranted equilibrium rate of growth:

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[1] It is, writes Harrod, 'that addition to capital goods in any period which producers regard as ideally suited to the output which they are undertaking in that period.' ('An Essay in Dynamic Theory', p. 260.)
[2] J.A. Kregel, *Rate of Profit, Distribution and Growth: Two Views* (Macmillan, London, 1971), p. 106.

$$Gw = \frac{s}{v_R}$$

This equation, says Hywel Jones, 'is no longer a truism. It expresses the rate of growth of output, which we denote Gw, which will satisfy entrepreneurs that they are investing the correct amount.'[1] To quote Harrod: 'I define Gw as that over-all rate of advance which, if executed, will leave entrepreneurs in a state of mind in which they are prepared to carry on a similar advance.'[2] It is a 'moving equilibrium, in the sense that it represents the one level of output at which producers will feel in the upshot that they have done the right thing, and which will induce them to continue in the same line of advance. Stock in hand and equipment available will be exactly at the level which they would wish to have them.'[3]

What Harrod's 'fundamental equation' Gw = s/v_R tells us is that the equilibrium rate of growth in a closed economy with no technical progress depends upon the exogenously determined share of output that is saved and invested (S/Y = s) and the technically defined quantity of capital necessary to increase output by a given amount (I/ΔY = v_R). The higher the share of national income saved and invested, and the lower the capital-output ratio (which is to say, the higher the Marginal Efficiency or productivity of Capital ΔY/ΔK), the higher will be the rate of equilibrium economic growth. For example, if the MPS = APS is 0.2, and the required capital/output ratio is 4, then the warranted rate of economic growth is:

Gw = $\frac{0.2}{4}$ = 0.05 = 5 per cent.

If the marginal and average propensity to save were 0.4, then:

Gw = $\frac{0.4}{4}$ = 0.1 = 10 per cent.

Given that s and v_R are assumed constant, the equilibrium rate of economic growth will be constant. Indeed, since I = ΔK, then:

I = S

can be written as:

ΔK = S

S = sY, so:

ΔK = sY

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[1] H. Jones, *Modern Theories of Economic Growth* (Nelson, London, 1975), p. 51.
[2] R.F. Harrod, *Towards a Dynamic Economics* (Macmillan, London, 1948), p. 82.
[3] Harrod, 'An Essay in Dynamic Theory', p. 264.

When we know the level of output Y and the capital-output ratio, then we know how much capital stock will be required for any given level of output. I.e.

$Y v_R = K$

In which case:

$$Y = \frac{K}{v_R}$$

Substituting this result into $\Delta K = sY$ gives us:

$$\Delta K = \frac{s}{v_R} K$$

From which it follows that:

$$\frac{\Delta K}{K} = \frac{s}{v_R}$$

Of course, s/v_R also determines the warranted rate of growth. Thus, in equilibrium, the rate of economic growth and the rate of increase in the capital stock must be equal. If the rate of increase in the capital stock per annum is 5 per cent, then the economy will be growing at 5 per cent per annum.

These relations suggest that an economy's rate of economic growth depends upon the share of GDP that is invested and the marginal productivity of capital. The higher the share of national income invested, or the higher the marginal productivity of capital, the faster will be the rate of economic growth. One reason China's economy currently grows at 5 per cent per annum and the UK's at 1.5 per cent is that China invests 40 per cent of its GDP and the UK invests 18 per cent.

The Actual and the Warranted Rate of Growth

G_W is the equilibrium 'warranted' rate of growth of total output – which is the rate of growth of output such that, given the required capital/output ratio, entrepreneurs want and do invest just the amount that households want to save, so that planned saving equals planned investment. As such, the warranted growth rate is dictated by the MPS and the required capital/output ratio. It is distinct from the *actual* growth rate, G_A, which is the rate of growth that actually occurs. We have seen that:

$G_A \equiv s/v_A$

So:

$G_A v_A = s$

Similarly, we know that $G_W = s/v_R$, so:

Gwv_R= s

It follows that:

G$_A$$v_A$ = s = Gwv_R

This means that, if the actual growth of output equals the warranted growth of output, then the actual ratio of capital to output must be equal to the ratio that entrepreneurs require as defined by the fixed capital/output ratio, and then planned investment equals actual investment and the economy is growing at the equilibrium steady state. This rate of growth can be expected to continue into the future. By contrast, if the actual rate of growth *exceeds* the warranted, then the actual capital/output ratio is *below* the required capital/output ratio and capitalists are investing *less* than they want to invest. Actual investment is less than planned investment and the growth rate is not at equilibrium. And if the actual rate of growth is *below* the warranted rate, then the actual capital/output ratio exceeds the required capital/output ratio and entrepreneurs are investing *more* than they want and plan to invest, so again the economy is not in equilibrium. The consequences of these divergences are considered below.

The Natural Rate of Growth

So far we have presented output to be a function of capital inputs only. However, Harrod recognised that production is a function of labour as well as capital inputs. Just as the capital/output ratio is given and fixed, so is the labour/output ratio. Let us define the fixed labour/output ratio as u_R. Thus, Yu_R = L or Y = $\dfrac{L}{u_R}$. Hence, Y is now a function of capital and labour, and we can write:

Y = min $\left[\dfrac{K}{v_R}, \dfrac{L}{u_R}\right]$

This means that output Y is a function of whichever is lower, K/v_R or L/u_R. In other words, if L/u_R is lower than K/v_R, and if u_R is fixed, then output is capped according to the amount of labour and the labour/output ratio and no amount of extra capital can raise this output. In this case there is no possibility of substituting capital for labour or *vice versa* and, in effect, capital and labour must be used in fixed proportions. An example would be pilots and planes: an airplane requires one pilot and one plane, and holding either fixed and increasing the other cannot yield an increase in flights. Output is constrained by whichever factor is in most short supply – so if you have 10 pilots and 20 planes, only 10 planes can be used and the other 10 must sit idle.

Harrod assumed that labour, rather than capital, was the constraint, so that total output per unit of time is determined by L/u_R. Over time, labour supply can be expected to increase, where the rate of increase in labour supply is:

$$\frac{\Delta L}{L} = n$$

n, the rate of increase of the labour supply, is determined exogenously, being determined by the rate of growth of population, and this is something determined outside the model. If we assume full employment and a fixed labour/output ratio, then in the long run the rate of growth of output cannot exceed the rate of growth of the labour supply:

$$\frac{\Delta Y}{Y} = \frac{\Delta L}{L} = n$$

Of course, if there is unemployed labour, then the natural rate of growth can temporarily exceed the rate of population growth – for example, growth may be faster after a recession.

This long-run growth rate of n makes no allowance for technical change. To include technical change, Harrod assumes that it occurs at an exogenously determined rate, with the result that output per head increases at a rate t, where t = ΔT/T. The long-run growth rate compatible with full employment and the growth in the labour supply and the given rate of technical change is the *natural* rate of growth, G$_N$. That is:

G$_N$ = n + t

This is the 'full employment' ceiling growth rate given labour supply growth and the rate of technical change. Hence, the actual rate of growth, G$_A$, cannot permanently exceed the natural rate of growth – though it can fall short of the natural rate, in which case not all labour will be employed. Thus:

G$_A$ ≤ n + t

So, for the economy to be growing at a steady-state equilibrium rate, where planned investment equals planned saving and all labour is employed, it must be growing at a rate where actual growth equals warranted growth equals the growth of the labour supply and technical change:

$$\text{G}_A = \frac{s}{v_A} = \text{G}_W = \frac{s}{v_R} = \text{G}_N = n + t = \frac{\Delta L}{L} + \frac{\Delta T}{T}$$

When the actual, warranted, and natural growth rates are all equal, then the economy will be in full employment equilibrium and growing at a constant rate equal to the rate of population growth. This has been called the 'Golden Age.' Unfortunately, like most Golden Ages, it is unlikely to exist: there is nothing in the Harrod model to ensure that actual growth (s/v_A) will equal warranted growth (s/v_R), or that either will equal the rate of labour-supply and productivity growth, n + t. In fact, it is very improbable. To quote Jones:

Although steady state growth at full employment is possible in a Harrod-type model of economic growth, such a 'Golden Age' is highly improbable given the independent constituent variables in the necessary equality of the warranted rate of growth, s/v_R, to the natural rate of growth, n. This conclusion is thoroughly 'Keynesian' in spirit: there is no reason to believe that full-employment equilibrium growth will be attained.[1]

One notable implication is that, given that for equilibrium full-employment growth G_W must equal G_N, then:
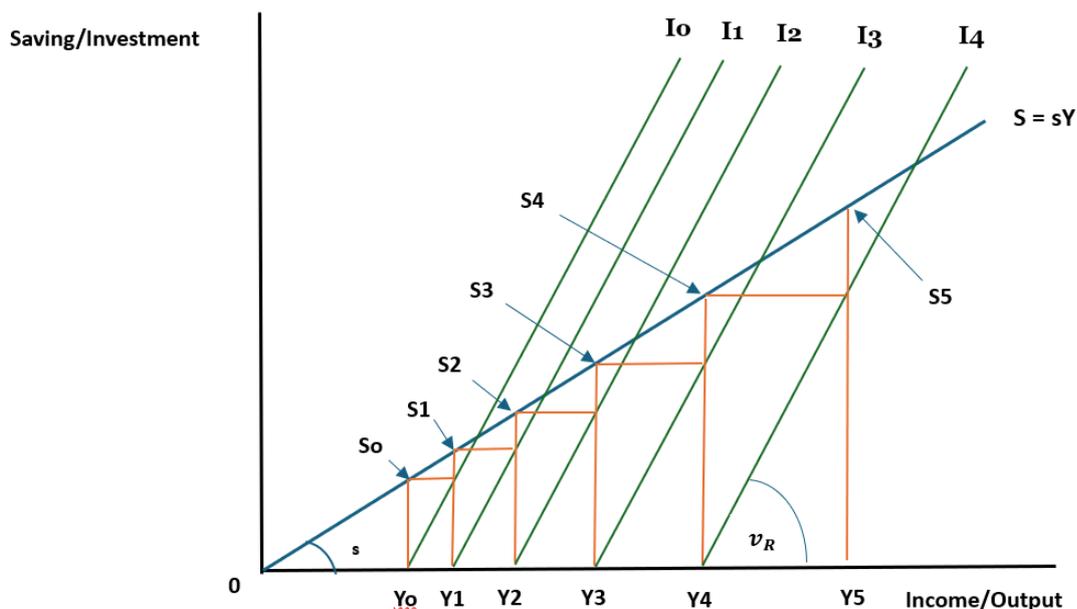
$$G_W = \frac{s}{v_R} = G_N = n + t$$

In this equation, *s*, n and t are all given, so if technical progress t is occurring then this cannot cause v_R to change. It, too, must remain constant. Hence, Harrod assumes that technical change is *neutral* in the sense that it leaves the output/capital ratio unchanged and doesn't cause firms to substitute capital for labour or labour for capital.

If the warranted growth rate G_W exceeds the natural growth rate G_N then the economy will tend towards full employment. Once this situation is reached there will be excess demand as the amount firms are trying to increase capacity by (s/v_R) will exceed the productive potential of the economy and there will be inflation and business investment plans will be frustrated due to a lack of manpower. The output constraint imposed by lack of labour will cause capital to be under-utilised and profit rates fall. Firms will cut back investment, unemployment will rise, and the economy will tend to depression. If, on the other hand, G_W is less than the natural growth rate then there will be unemployment and un-utilised technical advances. So, whenever the warranted rate diverges from the natural rate of growth, the result is a diminished rate of growth and unemployment.

The process of warranted equilibrium growth can be illustrated diagrammatically as follows.

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[1] Jones, *Modern Theories of Economic Growth*, p. 53.

Figure 1. Depiction of Harrod's Model of Warranted Equilibrium Growth

Figure 1 illustrates the process of equilibrium warranted growth in the Harrod model. The x-axis shows total real output or income, the y-axis total saving and investment. The saving function is S = sY, where s is the marginal propensity to save and determines the slope of the function. To each level of income, Yo, Y1, Y3 etc., there corresponds a planned level of total saving, So, S1, S2, as determined by the MPS. Total planned saving equals total planned investment. So when saving is YoSo, then investment is YoSo. Each level of additional investment increases output and income according to the required capital-output ratio v_R. The capital-output ratio is assumed to be greater than the MPS, hence the investment functions have a gradient v_R which is steeper than the gradient of the saving function. Given the required capital-output ratio, a level of saving and investment will yield a certain increase in real output, which is indicated by a rightward increase in income along the x-axis. For example, assume that the level of income and output is initially Yo. Given the MPS, income Yo corresponds to planned saving of YoSo. This saving is invested. Since planned investment equals planned saving, then, given the required capital-output ratio, output and income increase from Yo to Y1 – i.e. YoY1 is the increase in output generated by saving and investment SoYo, and is less that SoYo since the capital output ratio is greater than one. At the higher level of income Y1 saving increases to S1. This level of saving S1 is equal to planned investment and national income rises to Y2 as the increased capital investment increases the total output of the economy, again according to the constant required capital-output ratio. Income of Y2 yields saving and investment of S2, and income increases again to Y3. Total saving goes to S3Ys and when this is invested, income rises to Y4, and so on. The absolute increase in income steadily rises, but the rate of growth of income and output remains the same. This is

steady-state equilibrium warranted growth and can be maintained so long as the warranted growth rate is equal to the natural growth rate, GN.

Instability in the Harrod Growth Model: the 'Knife Edge' Problems

As we have noted, for long run growth to be stable at full employment in the Harrod system, then GA = GW = GN. This set of equalities is highly unlikely. But even if it is attained, any deviation between the actual and warranted growth rates can cause a divergence that increases explosively, with no prospect of a return to equilibrium. It is therefore possible to identify two 'knife-edge' problems in the Harrod model:

1. Will actual growth be equal to warranted growth? This is the question of equilibrium, which is to say, is planned investment equal to planned saving?
2. Will equilibrium warranted growth be equal to the natural rate of growth, which is to say, will the economy be at full employment in the long run?

The First Knife-Edge Problem: Actual Growth and Warranted Growth

One way to see this problem is as follows.

We know that, for equilibrium, an economy needs to be growing at its warranted rate:

GAv_A = s = GWv_R

where *s* is the fixed marginal/average propensity to save. Here, GA = GW *only if* the actual capital-output ratio equals the required capital-output ratio. But if the *actual* capital-output ratio (v_A) exceeds the required ratio (v_R), then actual growth GA will be lower than the warranted growth rate GW. And if the actual capital-output ratio is *less than* the required ratio then actual growth will exceed warranted growth. In either case the economy is *not* in its equilibrium growth path since firms are not investing what they need and plan to invest.

Suppose, for example, that firms become more pessimistic about future growth and decide to scale back their planned investment. Planned investment will now be below planned and actual saving. The immediate effect of reducing their purchases of investment goods is to cause the stocks of capital-producing firms to rise. With investment demand flagging, firms in general will find their stocks increasing. Additions to stocks are additions to capital, so, by cutting investment demand, v_A actually increases relative to v_R – firms are 'investing' too much relative to the output they are producing and more than they planned to invest (because of unplanned additions to stocks). Since v_A has risen, GA < GW. Firms respond by *scaling back their planned investment* again. However, by reducing their investment they will further slow the rate of growth of demand, causing another rise in unplanned additions to

stocks, pushing up v_A and lowering G$_A$. So firms cut investment again. Thus, G$_A$ declines continuously relative to G$_W$ and the economy will enter a depression.

By contrast, suppose firms become more optimistic about future business prospects and decide to invest more in new capital. Planned investment rises – but planned saving has not changed, so planned investment exceeds saving. It is saving that provides the goods for investment, so firms find that their planned investment is exceeding the resources available to build plant, factories etc. Since investment demand is rising, firms making investment goods will run down their stocks, and as the investment demand spreads through the economy, consumer-goods making firms will similarly deplete their stocks. As a result, the *actual* capital-output ratio will be below the *required* capital-output ratio. Since $v_A < v_R$ actual investment is below planned investment and actual growth will exceed warranted growth. In essence, firms are trying to invest too much in the sense of more than planned saving will allow. However, firms do not respond by reducing investment and actual growth back down to the warranted rate. Quite the reverse! Because stocks and the capital-output ratio are falling, firms respond by *increasing* their planned investment to replenish their stocks. 'Firms', says Harrod, 'finding themselves short of stock or equipment will increase their orders.'[1] This causes investment demand to further increase, driving up actual growth and reducing stocks and the capital-output ratio again, so firms will increase planned investment once more. Hence G$_A$ will rise continually relative to G$_W$. The economy is trying to grow beyond what can be sustained and the result is likely to be inflation and labour shortages.

Although Harrod generally reasons in terms of a fixed marginal propensity to save, in which case instability arises out of entrepreneur's fluctuating investment decisions, he notes that the same results obtain if households change their propensity to save. For example, if households become more pessimistic about future economic prospects they may plan to save more. In this case the MPS will rise. A rise in the planned value of *s* causes the warranted rate of economic growth to rise – given the fixed required capital-output ratio, the economy can grow faster. However, the immediate effect of the rise of planned saving is to cause the warranted rate of growth to exceed the actual rate of growth. Since households are saving more and spending less, firms find that demand is growing less rapidly than they anticipated and unsold stocks rise. The actual capital-output ratio rises above the required ratio. Firms respond by scaling back investment spending, causing demand to fall again and actual growth to decelerate further, with more upward pressure on stocks. Firms cut planned investment again and 'this depressing influence will drag the actual rate progressively farther below the warranted rate.'[2] Writing at the end of the 1930s, when unemployment and excess saving had characterised the economy, with actual rates

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[1] Harrod, 'An Essay in Dynamic Theory', p. 262.
[2] *Ibid*., p. 264.

of growth below what was warranted, led Harrod to believe such a depressive scenario to be likely.

In Harrod's basic model, however, the marginal propensity to save is fixed. This means that variations in demand causing actual growth to diverge from warranted growth are likely to arise from variations in investment. As we have seen, a rise in actual growth above the warranted growth rate is likely to arise if firms optimistic 'animal spirits' cause them to *expect* economic growth to exceed the warranted rate. This causes them to raise their investment plans above planned saving and the economy is no longer in equilibrium. Aggregate demand rises and inventories decline. As a result, the capital-output ratio will fall below the required rate. Thus, it is business-people's *expectations* regarding future growth that is central to Harrod's 'knife edge' stability problem. This aspect of the model can be explored using the analysis of Amartya Sen, which highlights the importance of business expectations in forming the trajectory of the economy.[1]

First, we outline some definitions:

Y_t^E = The income and output expected by entrepreneurs in time period t.

Y_t = The actual level of output and income in period t.

G_t^E = The expected rate of output growth between period t-1 and t. It is defined as follows:

$$G_t^E = \frac{Y_t^E - Y_{t-1}}{Y_t^E}$$

This equation says that firms expect output in period t to be some amount higher than output in t-1, with the difference expressed relative to the output expected in period t.

G_t = The actual rate of growth of income between periods t-1 and t. It is calculated as follows:

$$G_t = \frac{Y_t - Y_{t-1}}{Y_t}$$

Using the conventional Keynesian multiplier theory, we can determine national income in period t as:

$$Y_t = \frac{1}{s} I_t$$

Here, *s* is the marginal propensity to save, hence 1/*s* is the multiplier, and I_t is the level of investment in period t.

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[1] I have adapted this exposition from Jones, *Modern Theories of Economic Growth*, pp. 54-57.

Investment is determined *via* the *accelerator principle*, so that the investment firms *plan* to undertake in period t depends on the expected change in income and the required capital-output ratio, v_r.

$$I_t = v_r(Y_t^E - Y_{t-1})$$

In other words, a business-person who expects output to increase a certain amount between period t-1 and t will need to invest in capital, the amount of extra investment required depending on the capital-output ratio.

Substituting the investment equation into the income equation we get:

$$Y_t = \frac{v_r}{s}(Y_t^E - Y_{t-1})$$

Dividing both sides by Y_t^E we get:

$$\frac{Y_t}{Y_t^E} = \frac{v_r}{s}\left(\frac{Y_t^E - Y_{t-1}}{Y_t^E}\right)$$

The expression in parentheses is, as we have seen, the expected rate of growth, G_t^E. Hence, we can re-write this equation as:

$$\frac{Y_t}{Y_t^E} = \frac{v_r}{s} \cdot G_t^E$$

Now, if *expected* income in period t (which is Y_t^E) is equal to *actual* income in t (i.e. Y_t), then business expectations are realised. In this case, $Y_t/Y_t^E = 1$ and hence:

$$1 = \frac{v_r}{s} \cdot G_t^E$$

It follows that:

$$G_t^E = \frac{1}{v_r/s} = \frac{s}{v_r}$$

This is an important result. It tells us that when entrepreneurs' expectations regarding total output or income in period t are realised, then the expected rate of growth (G_t^E) will be equal to s/v_r, where s/v_r is our old friend, the *warranted rate of economic growth*. So, when firms expect the rate of economic growth to equal the warranted rate of economic growth, then their expectations will be realised, planned investment will equal planned saving, and the economy will be on a constant steady-state growth path.

Problems begin, however, when entrepreneurs expect a rate of economic growth that does *not* equal the warranted rate. If the expected rate of growth exceeds the warranted rate, then actual growth will exceed both; and if expected growth is less

than the warranted rate, then actual growth will be lower than both. We can prove these results as follows.

We return, first, to the equation determining actual income compared to expected income and derive expressions for Y_t and Y_t^E.

$$\frac{Y_t}{Y_t^E} = \frac{v_r}{s} \cdot G_t^E$$

We know that:

$$G_t^E = \frac{Y_t^E - Y_{t-1}}{Y_t^E}$$

This can be re-written as:

$$G_t^E = \frac{Y_t^E}{Y_t^E} - \frac{Y_{t-1}}{Y_t^E}$$

$$G_t^E = 1 - \frac{Y_{t-1}}{Y_t^E}$$

$$\frac{Y_{t-1}}{Y_t^E} = 1 - G_t^E$$

$$Y_{t-1} = Y_t^E - Y_t^E G_t^E$$

$$Y_{t-1} = Y_t^E (1 - G_t^E)$$

$$Y_t^E = \frac{Y_{t-1}}{1 - G_t^E}$$

Similarly, given that actual growth is:

$$G_t = \frac{Y_t - Y_{t-1}}{Y_t}$$

then:

$$G_t Y_t = Y_t - Y_{t-1}$$

$$G_t Y_t - Y_t = - Y_{t-1}$$

$$Y_t (G_t - 1) = - Y_{t-1}$$

$$Y_t = \frac{- Y_{t-1}}{G_t - 1}$$

$$Y_t = \frac{Y_{t-1}}{1 - G_t}$$

We can now insert these expressions for Y_t and Y_t^E into:

$$\frac{Y_t}{Y_t^E} = \frac{v_r}{s} \cdot G_t^E$$

$$\frac{\frac{Y_{t-1}}{1-G_t}}{\frac{Y_{t-1}}{1-G_t^E}} = \frac{v_r}{s} \cdot G_t^E$$

$$\frac{Y_{t-1}}{1-G_t} = \frac{v_r}{s} \cdot G_t^E \left(\frac{Y_{t-1}}{1-G_t^E}\right)$$

$$Y_{t-1} = \frac{v_r}{s} \cdot G_t^E \left(\frac{Y_{t-1}}{1-G_t^E}\right)(1-G_t)$$

$$Y_{t-1} = \frac{v_r}{s} \left(\frac{G_t^E Y_{t-1}}{1-G_t^E}\right)(1-G_t)$$

$$\frac{Y_{t-1}}{\frac{v_r}{s}\left(\frac{G_t^E Y_{t-1}}{1-G_t^E}\right)} = (1-G_t)$$

$$G_t = 1 - \frac{Y_{t-1}}{\frac{v_r}{s}\left(\frac{G_t^E Y_{t-1}}{1-G_t^E}\right)}$$

$$G_t = 1 - \frac{s}{v_r}\left(\frac{1-G_t^E}{G_t^E Y_{t-1}}\right)Y_{t-1} \qquad [Y_{t-1} \text{ cancels out}]$$

$$G_t = 1 - \frac{s}{v_r}\left(\frac{1-G_t^E}{G_t^E}\right)$$

This equation determines the *actual* rate of economic growth. Remember that s/v_r is the *warranted* rate of economic growth. What this equation tells us is that the expected rate of growth will equal the actual rate of economic growth *only if* the expected rate of growth equals the warranted rate of economic growth. To see this, assume that expected growth does equal the warranted rate of growth so $s/v_r = G_t^E$. Then:

$$G_t = 1 - \frac{s}{v_r}\left(\frac{1-\frac{s}{v_r}}{\frac{s}{v_r}}\right)$$

$$G_t = 1 - (1 - \frac{s}{v_r}) = \frac{s}{v_r}$$

As can be seen, when expected growth equals warranted growth, then actual growth will equal warranted growth. When this is true, actual, expected, and warranted growth are all equal and steady-state equilibrium growth is achieved. But if, for example due to an uplift in entrepreneurs 'animal spirits', firms expect economic growth to *exceed* the warranted rate of growth ($G_t^E > s/v_r$) then, not only will *expected* growth exceed *warranted* growth, but *actual growth will exceed expected growth.* And if firms become

more pessimistic and expect growth to be *lower* than the warranted rate of growth (G_t^E < s/v_r), then not only will expected growth be lower than warranted growth, but *actual growth will fall short of expected growth*. So: when firms are optimistic, actual growth will *exceed* expectations, and when firms are pessimistic, actual growth will be even slower than expected.

To illustrate. Suppose the marginal and average propensity to save (*s*) is 0.2 and the capital/output ratio (v_r) is 2. Then the warranted rate of growth will be:

$$G_W = \frac{s}{v} = \frac{0.2}{2} = 0.1 \text{ or 10 per cent.}$$

If entrepreneurs expect growth to be 10 per cent, then actual growth will be:

$$G_t = 1 - \frac{s}{v_r}\left(\frac{1 - G_t^E}{G_t^E}\right) = 1 - \frac{(1-0.10)}{0.10}0.1 = 1 - \frac{0.9}{0.1}0.1 = 1 - (9)(0.1) = 1 - 0.9 = 0.10 = 10\%$$

Thus, if entrepreneurs expect growth to be at the warranted rate, then actual growth will equal warranted growth and the economy will expand at the warranted rate. Planned saving and planned investment will be equal. But, if entrepreneurs *expect* growth to be 11 per cent, then actual growth will be:

$$G_t = 1 - \frac{s}{v_r}\left(\frac{1 - G_t^E}{G_t^E}\right) = 1 - \frac{(1-0.11)}{0.11}0.1 = 1 - \frac{0.89}{0.11}0.1 = 1 - (8.1)(0.1) = 1 - 0.81 = 0.19 = 19\%$$

Thus, actual growth will be 19 per cent – significantly higher than the warranted rate of 10 per cent and the expected rate of 11 per cent. Since actual growth exceeds expectations, entrepreneurs will revise upward their expectations of future growth, and growth will accelerate further and further away from the warranted rate of growth. There is no self-correcting mechanism and the rate of growth is highly unstable. If, on the other hand, firms expect growth to be *lower* than the warranted rate of 10 per cent, say 9 per cent, then actual growth will be:

$$G_t = 1 - \frac{(1-0.09)}{0.09}0.1 = 1 - \frac{0.91}{0.09}0.1 = 1 - (10.11)(0.1) = 1 - 1.01 = -0.01 = -1\%$$

In this case, actual growth will be negative and below what entrepreneurs had expected. Pessimism will yield a severe recession, firms revising downwards their expectations of future growth, cutting investment back, and the economy will tend towards a slump. Because firms invested too little, they have ended up producing too much and cannot sell their output – hence Harrod's famously paradoxical remark: 'Over-production is the consequence of production below the warranted level.'[1]

Harrod concludes as follows:

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[1] Harrod, 'An Essay in Dynamic Theory', p. 266.

The dynamic theory so far stated may be summed up in two propositions. (i) A unique warranted line of growth is determined jointly by the propensity to save and the quantity of capital required by technological and other considerations per unit increment of total output. Only if producers keep to this line will they find that on balance their production in each period has been neither excessive nor deficient. (ii) On the other side of this line is a 'field' in which centrifugal forces operate, the magnitude of which varies directly as the distance of any point is from the warranted line. Departure from the warranted line sets up an inducement to depart farther from it. The moving equilibrium of advance is thus a highly unstable one.[1]

This has come to be called the first knife-edge problem.

The Second Knife-Edge Problem: Warranted and Natural Growth Rates

Harrod's second knife-edge problem related to the natural growth rate and the warranted growth rate. The natural growth rate is exogenously determined by the rates of growth of population and technology ($G_N = n + t$). Since there is only one natural growth rate there will be only one warranted growth rate corresponding to it. If the warranted rate, which is the equilibrium rate where planned investment by firms equals planned saving, exceeds the natural rate, then firms investment plans cannot fully realised since the growth of capital cannot exceed the growth in the labour supply. Built capital will not be fully utilised and, with capital sitting idle, profits will be less than anticipated. The capital-output ratio will exceed the warranted ratio and actual growth will be below the warranted level. Firms respond by cutting back investment plans, demand will fall, and a downward spiral of the sort analysed above will be initiated. On the other hand, if warranted equilibrium growth is below the natural rate, then there will be unemployed labour and unexploited technology and the economy will fall short of its long-run growth potential.

This is the second Harrod 'knife-edge' problem. It is quite likely to arise since, as Harrod observes, there is 'no inherent tendency' for the warranted and natural rates of growth to coincide.[2] This opened the possibility of government policy to seek to bring the warranted growth rate closer to the natural. For example, if the warranted rate exceeds the natural then interest rate policy might be deployed: a lowering of the rate of interest might raise the capital-output ratio (as less productive investment projects would be rendered profitable) and lower the marginal propensity to save, both of which would lower the warranted rate of growth.

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[1] *Ibid*., p. 265.
[2] *Ibid*., p. 273.

Criticisms of Harrod's Growth Model

Being a pioneer excursion into dynamic growth theory, and one generating some rather startling results, Harrod's work has been subject to numerous critical assessments.[1] Among the notable critical objections have been the following:

I.   *Harrod's model contains too many exogenously determined variables which are not explained.* Saving is determined simply by the marginal propensity to save and the level of income. It is taken as given and does not reflect the influence of other variables, such as the rate of interest or the distribution of income. The capital and labour-output ratios are assumed constant and there is no allowance for variations in either in response to relative factor prices, interest rates, or technical change. The rate of technical change is given, as is the rate of population growth. In a growth model purporting to explain change over time, such rigidity is a serious weakness: whilst growth occurs over time, most of the key variables are a-temporal – which is not realistic. Harrod remains, in many ways, within a comparative static paradigm.

II. *The model is unrealistically unstable.* Lacking any mechanism for ensuring that actual growth will equal equilibrium growth, it is highly likely the two will diverge, and once they do, the effects are explosive. Steady-state growth appears, from Harrod's work, to be something unexpected – yet, the post-war experience has shown it to be much closer to the norm. Actual growth has not been particularly unstable, and there has not been the tendency to stagnation and unemployment that Harrod considered most likely.

III. *Harrod's theory of investment is too simplistic.* Investment is chiefly determined through the accelerator model. But investment clearly also depends on other variables, such as the rate of interest, business expectations, and the rate of profit. These are either not included in the model or not explained. Harrod's modelling of expectations is weak. Business optimism or pessimism is not accounted for, and business-people don't learn from experience: if they find that that their investment plans exceed actual investment, they respond by continually increasing investment, causing actual growth to exceed warranted growth by larger and larger amounts. It never occurs to them to scale-back their investment plans. It is this unquestioning tendency to extrapolate which causes the system to be so unstable.

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[1] Jones, *Modern Theories of Economic Growth*, pp. 65-68 and Kregel, *Rate of Profit, Distribution and Growth*, pp. 113-118, both provide a review of criticisms of the model.