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Robert Solow's Economic Growth Model

Dr Ian St John

Economics Department

istjohn@habselstree.org.uk

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Abstract

Robert Solow's 1956 paper on economic growth has powerfully shaped thinking about long run economic growth. Here we expound Solow's ideas, focusing on the roles of investment per worker and depreciation, population growth, and technical change in determining balanced growth in per capita GDP. While capital depreciation and population growth tend to decrease capital and output per worker, investment and technical progress tend to raise it. Increasing saving and hence investment as a share of GDP raises output per worker, but it also shifts resources from consumption to investment goods. Hence, we also explore the Golden Rule level of capital worker which maximises consumption per worker. Empirical studies suggest that in the case of both the United States and China investment per worker is below Golden Rule levels, though in the case of China high levels of investment per worker are moving towards Golden Rule consumption maximising levels. Over time the Solow model predicts that economic growth rates will converge towards similar rates between countries.

Robert Solow's highly influential model of long run economic growth was published in 1956 as 'A Contribution to the Theory of Economic Growth.'[1] Building on Roy Harrod's 'An Essay on Dynamic Theory', it followed Harrod in assuming a fixed ratio of saving to income and that all saving is automatically invested, but combined this with a neo-classical production function with varying ratios of capital to labour. Here we outline its chief features and determine the 'Golden Rule' level of capital accumulation.[2]

[1] R.M. Solow, 'A Contribution to the Theory of Economic Growth', *The Quarterly Journal of Economics* (Vol. 70, No. 1, February 1956, pp. 65-94. Solow provided an expanded discussion in, R.M. Solow, *Growth Theory: An Exposition* (Clarendon Press, Oxford, 1970).

[2] In developing this model I have especially drawn upon N.G. Mankiw, *Macroeconomics* (Worth Publishers, New York, Ninth Edition, 2016), O. Blanchard, A. Amighini, and F. Giavazzi, *Macroeconomics: A European Perspective* (Pearson, London, Second Edition, 2013), S.D. Williamson, *Macroeconomics* (Person, Third Edition, 2008), D. Romer, *Advanced Macroeconomics* (McGraw-Hill,

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Assumptions of the Model

The economy makes just one product. This is in the tradition of classical 'corn' models, where corn is the only output and, as seed, represents capital investment too. Hence the total output of this one product is the total output of the economy – what we may call GDP. Since there is only one product there were no relative prices, so the price of this product can be set at 1.

We assume a closed economy with no trade.

The factor inputs are capital and labour and technology. Land and natural resources are not a factor of production.

There is no government, so there is no government spending or taxation.

Initially population is assumed to be constant and all people are workers, which means the labour supply L is fixed. As we develop the model we will drop this assumption and have population and labour supply increase by an exogenously determined rate, n.

Technology is taken to be exogeneous to the firm. It is given to the firm and does not depend on its own research and development. To begin with we assume that the level of technology is fixed. Later we will assume it to be variable but still exogenous.

Solow's model operates through the interaction of two functions – a neoclassical production function and a Keynesian demand function.

The Production Function

A production function maps output to inputs. Output (Y) for Solow is a function of the inputs of capital (K) and labour (L). So:

$$Y = F(K, L) \qquad (1)$$

Solow assumes constant returns to scale, so if you increase both inputs by a common factor (λ) then output will increase by that common factor too. Thus:

$$\lambda Y = F(\lambda K, \lambda L)$$

So if $\lambda = 2$, then doubling inputs of capital and labour will cause total output to double also.

Solow is chiefly concerned with output per worker. To arrive at this, we divide (1) by the labour input (L) to get the 'intensive form' of the production function:

$$\frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \qquad (2)$$

New York, Third Edition, 2006), and R.J. Barro, *Macroeconomics* (MIT Press, Cambridge, Fifth Edition, 1997).

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Thus, output per worker (or labour productivity) depends on capital per worker (K/L) – technology remaining constant.

We designate output per worker as y and capital per worker as k.

$$y = \frac{\mathrm{Y}}{\mathrm{L}} \tag{3}$$

$$k = \frac{\mathrm{K}}{\mathrm{L}} \tag{4}$$

Hence we can now say that output per worker (y) is a function of capital per worker (k):

$$y = \mathrm{f}(k) \tag{5}$$

This is the basic production function assumed. As capital per worker increases, so will output per worker. In other words:

$$y' = f'(k) > 0 \tag{6}$$

This expression is the marginal product of capital – it means that as capital per worker k increases then output per worker y increases. We also assume diminishing returns, so that:

$$y'' = f''(k) < 0 \tag{7}$$

This means that as capital per worker rises output per worker rises but at a diminishing rate. When capital per worker is low, then an addition to the amount of capital each worker has will make a significant impact to their output; but when workers have a lot of capital already, an increase in capital per worker will have a small impact on total output. This production function can be depicted as follows.

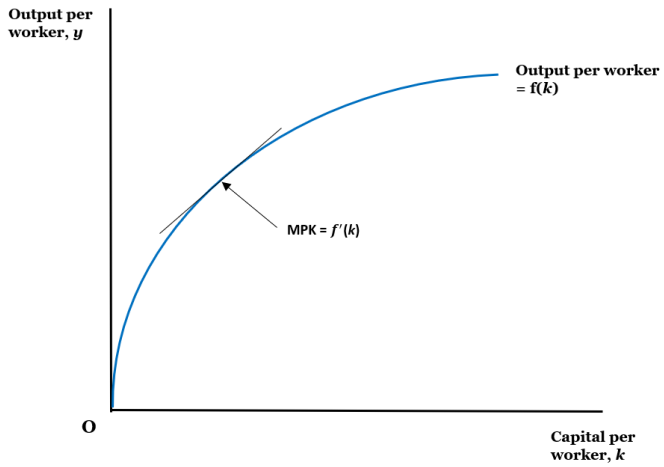


Figure 1. The Production Function: Output per Worker as a function of Capital per Worker

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A Cobb-Douglas Production Function

In his article Solow applied his model to the example of a Cobb-Douglas production function with constant returns to scale and this is the conventional way of illustrating his ideas. In this case the production function is:

$$\mathrm{Y} = \mathrm{K}^{\alpha}\mathrm{L}^{1-\alpha} \qquad (8)$$

Here the exponent α is $0 < \alpha < 1$ and since $\alpha + (1 - \alpha) = 1$ there are constant returns to scale.

To derive a per-worker expression, we divide the production function through by L to get:

$$\frac{\mathrm{Y}}{\mathrm{L}} = \frac{\mathrm{K}^{\alpha}\mathrm{L}^{1-\alpha}}{\mathrm{L}} = \mathrm{K}^{\alpha}\mathrm{L}^{-\alpha} = \frac{\mathrm{K}^{\alpha}}{\mathrm{L}^{\alpha}} = \left(\frac{\mathrm{K}}{\mathrm{L}}\right)^{\alpha}$$

We have already defined Y/L as y and K/L as k. Hence:

$$\frac{\mathrm{Y}}{\mathrm{L}} = \left(\frac{\mathrm{K}}{\mathrm{L}}\right)^{\alpha} \text{ becomes } y = k^{\alpha} \qquad (9)$$

This is our Cobb-Douglas production function. Note that it has diminishing return to capital per worker as:

$$\frac{dy}{dk} = \alpha k^{\alpha-1} \qquad (10)$$

Since $\alpha<1$, then the exponent of k will be negative, so as k increases the marginal product of k will decline.

Demand

In Solow's model there are two types of spending – consumption spending by households and investment spending by firms. This is the familiar Keynesian identity:

$$\mathrm{Y} = \mathrm{C} + \mathrm{I}$$

To put this in per-worker terms, we divide throughout by L to get:

$$y = c + i \qquad (11)$$

This means that output per worker (y) is accounted for as either consumption per worker (c) or investment per worker (i). The consumption behaviour of households is modelled along the lines of the Keynesian consumption function. People save a fraction of their income (s) and consume the rest (1-c). Thus the consumption function is:

$$c = (1 - s)y \qquad (12)$$

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These, again, are per-worker values. The saving rate or marginal propensity to save is between 0 and 1. Given that:

$y = c + i$

and $c = (1 - s)y$

then:

$y = (1 - s)y + i$

$y = y - sy + i$

$sy = i$ (13)

Thus saving per worker is equal to investment per worker. Solow assumed that whatever is saved by workers is invested and the rate of saving is equal to the fraction of output per worker that is invested. Investment per worker (i) is proportional to saving per worker.

These production and consumption functions define an economy at any moment in time. For a given capital stock per worker k, the production function determines the output of the economy:

$y = \mathrm{f}(k)$ (5)

while the savings rate (s) determines the allocation of that output between consumption and investment:

$y = (1 - s)y + i$ (14)

Steady State Growth

Given that technology is initially fixed, output per worker depends only on capital per worker, k. Labour supply is fixed, but the quantity of capital and hence capital per worker is not fixed. Two things cause it to change:

1. Depreciation

Depreciation is the process whereby capital goods wear out. In itself this causes capital per worker to decline. We assume that a certain proportion (δ) of capital wears out each year on average. This is the *depreciation rate*. For example, if 5% of capital wears out each year, then the depreciation rate is $\delta = 0.05$. The amount of capital per worker that wears out each year is then: δk. This is the rate at which depreciation *reduces* capital per worker.

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2. Investment

While depreciation reduces capital per worker, investment increases it. Investment per worker $i = sy$. Since $y = \mathrm{f}(k)$, then we can write:

$$i = s\mathrm{f}(k) \qquad (15)$$

This equation states that, for any value of k, then y is determined, and any value of y in conjunction with s determines the level of investment per worker – where investment *increases* the capital stock.

Thus, both output per worker and investment per worker are functions of capital per worker. Plotting both functions graphically yields the following diagram.

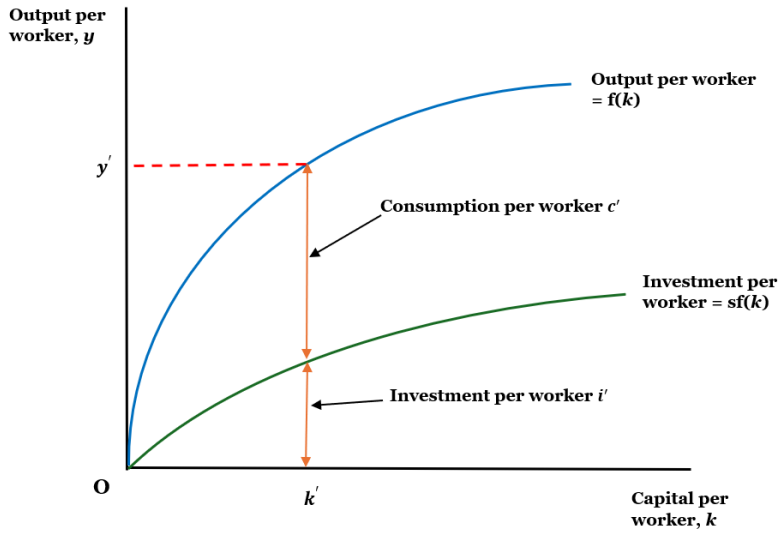


Figure 2. How Investment and Output Per Worker Determine Consumption

The production function showing output per worker as a function of capital per worker and the investment per worker line (which depends on saving per worker) each have the same shape – only the investment per worker line is lower by the factor s which is the saving rate. In other words, investment per worker is a constant fraction s of output per worker. Thus, when capital per worker is k', output per worker is y'. This output per worker is distributed between consumption per worker c' and investment per worker, i'. The distribution of output between investment and consumption depends on the level of output per worker and the marginal propensity to save: the higher the marginal propensity to save, the higher investment per worker at any given level of income per worker (since Solow assumes that saving is always equal to investment) and the lower the consumption per worker.

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The Steady State

As we have noted, over a given period of time, depreciation reduces the capital stock while investment increases it. The change in the capital stock per worker is then the difference between the two:

$$\Delta k = i - \delta k = sf(k) - \delta k \qquad (16)$$

Thus, what happens to capital per worker k depends on investment per worker and the effect of depreciation on capital per worker. If investment in capital per worker exceeds depreciation per worker then capital per worker will rise and so will output per worker since $y = f(k)$; if depreciation per worker exceeds investment per worker the capital per worker and output per worker will fall. If they are equal then capital per worker (k^*) and hence output per worker (y^*) will remain constant. This is the *Steady State*:

$$sf(k^*) = \delta k^* \qquad (17)$$

$$y^* = f(k^*) \qquad (18)$$

where k^* and y^* are steady state levels of capital per worker and output per worker respectively.

We have already plotted investment per worker as a function of capital per worker. We can also plot depreciation per worker as a linear function of the capital per worker since depreciation is a constant percentage of the capital stock: so as capital per worker rises so does the absolute amount of depreciation per worker.

Combining these two functions yields the following key diagram of Solow's model.

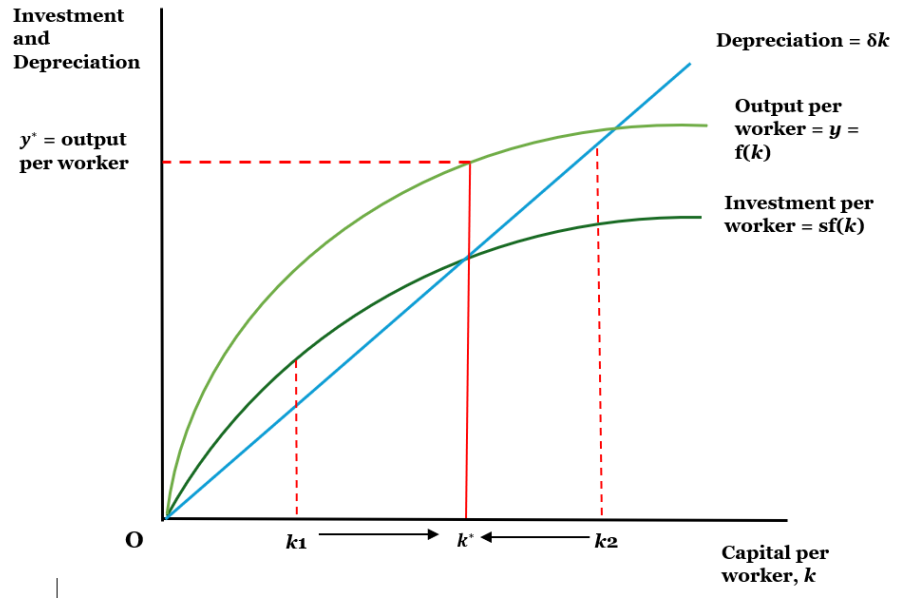


Figure 3. Determining the Steady State Capital and Output per Worker

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This diagram illustrates the tendency of an economy to a 'steady state' where the level of capital per worker is constant at k^* and output per worker is constant at y^*. At k^* the level of investment per worker is equal to the level of depreciation per worker, so the capital stock per worker is constant. An economy is either at the steady state level of capital per worker or tending towards it. If the capital stock per worker were *below* k^*, say at $k1$, then investment per worker exceeds capital depreciation per worker and capital per worker will *rise* until it arrives at k^*. And if capital per worker exceeds the steady state level k^* then the depreciation of the capital stock will exceed additions to the capital stock and the capital stock per worker will *decline* until it arrives at k^*. The long run equilibrium capital stock per worker is thus k^* and having reached this point the capital stock per worker will have no tendency to change and neither will output per worker – hence it is a 'steady state.'

Some Comparative Static Results

Suppose that in a society the average propensity to save rises. For some reason, people decide to forgo consumption and increase the share of income that they save. Since income that is not saved is invested, the result is an increase in investment per worker. What does this do to the economy?

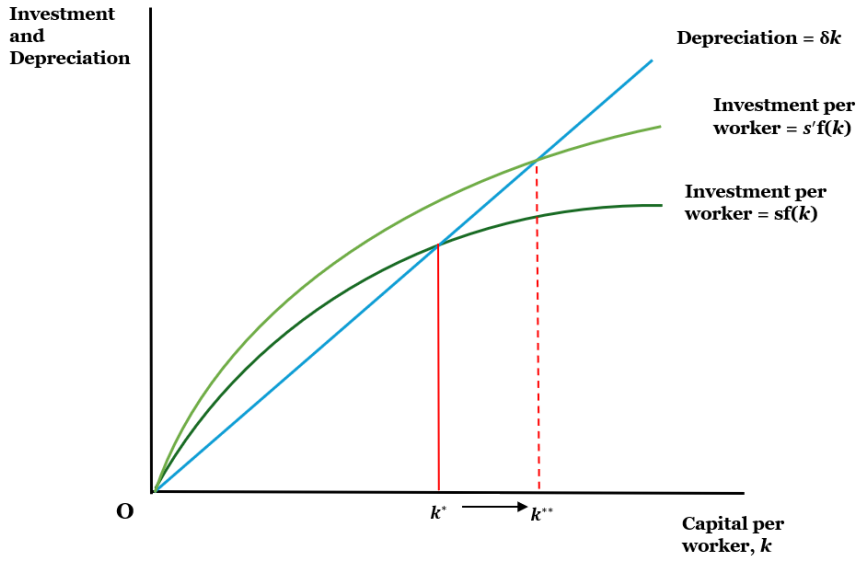


Figure 4. How a Rise in the Saving Rate Raises the Steady State Level of Capital per Worker

The above diagram shows how an increase in the saving and investment rate raises the equilibrium 'steady state' level of capital per worker. Initially the economy is in equilibrium at $k^*$with depreciation equal to investment, where investment is $\mathrm{sf(k)}$ given a saving rate s. The saving rate increases from s to s', which means that at the initial level of capital per worker (k^*) investment per worker exceeds depreciation per worker.

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The capital stock per worker rises. As capital stock per worker rises so does output per worker y. There is economic growth and rising incomes, which causes saving and investment to rise still further. However, due to diminishing returns, the rate of growth of income per worker and hence savings per worker diminishes as k increases. At the same time as the capital stock increases so does the absolute amount of depreciation. Eventually, at output per worker k^{**}, depreciation per worker is again equal to investment per worker and the economy arrives at a new steady state. At this point, capital per worker and hence output per worker stops growing. Thus, a one-off increase in the savings rate will produce a one-off increase in the capital/labour ratio and per capita GDP as the economy moves to its new equilibrium position. If, on the other hand, the saving and investment rate were to fall then the investment line would pivot down, depreciation would exceed investment at the initial steady state, and capital per worker and output per worker would decline until a new lower capital per worker and output per worker point is attained.

Assume, instead, that the rate of depreciation increases. Perhaps due to AI existing machines and computer systems become obsolete faster. A rise in the rate of depreciation δ will cause the depreciation line δk to pivot up. At the initial steady-state point k^* the capital stock is scrapped faster than investment adds to it. Capital per worker and output per worker fall. As capital per worker falls the absolute amount of depreciation declines until a new steady-state capital per worker is arrived at with a lower level of output per worker. The economy has contracted. This is shown in **Figure 5**.

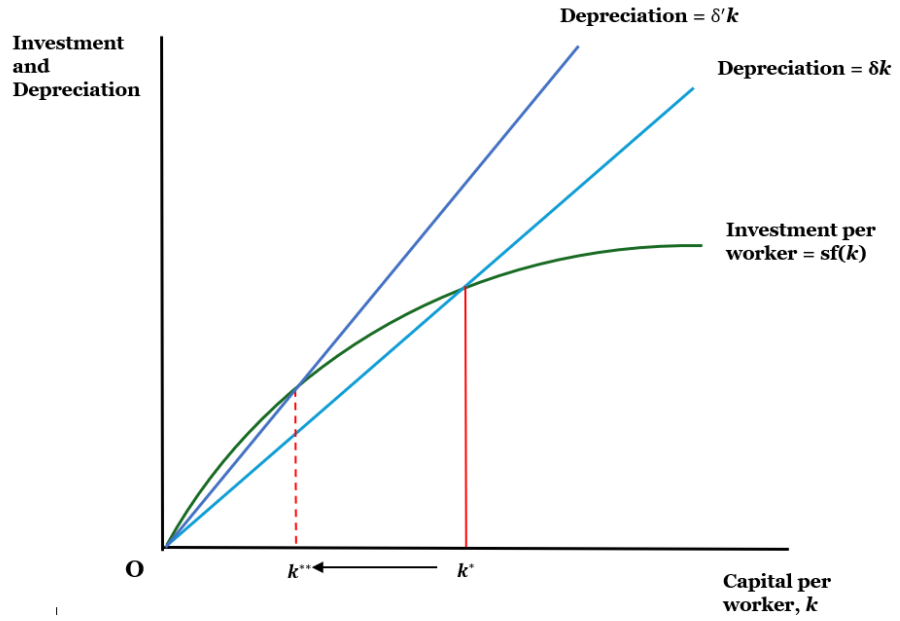


Figure 5. The Effect of a Rise in the Depreciation Rate on Steady-State Capital Per Worker

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In this diagram, as the rate of depreciation increases to δ' the steady-state level of capital per worker contracts from k^* to k^{**} with a correspondingly lower level of output per person.

We can illustrate these results with respect to a Cobb-Douglas production function. Suppose that:

$$\mathrm{Y} = \mathrm{K}^{0.5}\mathrm{L}^{0.5} \qquad (19)$$

Here total output is a function of the inputs of capital and labour with constant returns (since 0.5 + 0.5 = 1). In terms of per worker values we have:

$$\frac{\mathrm{Y}}{\mathrm{L}} = \frac{\mathrm{K}^{0.5}\mathrm{L}^{0.5}}{\mathrm{L}} = \frac{\mathrm{K}^{0.5}}{\mathrm{L}^{0.5}}$$

Since $y = \mathrm{Y}/\mathrm{L}$ and $k = \mathrm{K}/\mathrm{L}$, this can be written as:

$$y = k^{0.5} \qquad (20)$$

For a steady state to exist, investment in new capital per worker must equal depreciation of capital per worker:

$$\mathrm{s}y = \delta k^*$$

$$\mathrm{s}(k^*)^{0.5} = \delta k^*$$

Given this, then:

$$\frac{s}{\delta}(k^*)^{0.5} = k^*$$

$$\left(\frac{s}{\delta}\right)^2 k^* = (k^*)^2$$

$$\left(\frac{s}{\delta}\right)^2 = \frac{(k^*)^2}{k^*}$$

$$k^* = \left(\frac{s}{\delta}\right)^2 \qquad (21)$$

Thus the steady state level of capital per worker equals the square of the ratio of the savings rate to the depreciation rate. Other things being equal, if the savings rate rises the steady state level of capital per worker will rise; and if the depreciation rate rises, the steady state level of capital per worker will fall.

To summarise. For a given rate of depreciation, the equilibrium level of capital per worker in an economy depends on the saving and investment rate. The higher is the rate of saving and investment, the higher will be capital per worker and the higher output per worker. It follows that countries with a higher savings rate will tend to have a higher level of GDP per person.

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The 'Golden Rule' Level of Capital Per Person

We have seen that the higher the savings and investment rate in an economy, the higher will be the level of GDP per person. Yet this does not mean that, viewed in terms of social welfare, an economy with a higher GDP per person is better situated than one with a lower. The well-being of an economy's population depends upon the amount of goods and services those people can access. An economy doesn't produce for the sake of production – it produces for the sake of consumption. Here we face a trade-off: a country with a high rate of saving will tend to have a high level of output per person; but to secure this output, that society must save and not consume a larger share of what it produces. Similarly, a country which saves little and consumes more will enjoy more consumption in the short run – but at the cost of a lower GDP in the long run. What, then, is the optimal balance between saving per person and output per person? In terms of the Solow model, the optimum rate of saving is that rate which maximises the well-being of the population. We know that according to the Solow model any economy will automatically tend to a certain 'steady state' level of capital per worker k. That level of 'steady state' capital per worker which maximises consumption is called the *Golden Rule level of capital per worker* and we designate this as k^G.

We have seen already that output per worker (y) can be disaggregated into consumption per worker (c) and investment per worker (i). Thus:

$$y = c + i \qquad (11)$$

Hence consumption per worker is:

$$c = y - i \qquad (22)$$

We know that in a steady state investment per worker is equal to depreciation per worker so that net capital per worker remains constant:

$$i = \delta k^* \qquad (23)$$

This steady state level of capital per worker is $k^*$and this gives us the steady state output per worker $y^* = \mathrm{f}(k^*)$. If we now substitute these steady-state values back into our consumption equation we get the steady state level of consumption per worker:

$$c^* = y^* - i^*$$

$$c^* = \mathrm{f}(k^*) - \delta k^* \qquad (24)$$

This equation tells us that the steady state level of consumption per person is equal to the steady state level of output per person minus the steady state rate of depreciation. Economically this is intuitive: of the output per worker a certain amount must be set aside to cover depreciation (if it is not then machines will wear out and the capital-output ratio will fall and this is not a steady state) and what is then left over is available

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to be enjoyed as consumption. We can see the trade-off between capital per worker and consumption again: if the steady state capital per worker (k^*) rises then steady state output per worker (y^*) rises making possible *more* consumption – but, at the same time, as capital per worker rises so the amount which must be set aside to cover depreciation (δk^*) rises and this acts to *reduce* output available for consumption. The 'Golden Rule' level of capital per worker is that which maximises the positive difference between $f(k^*)$ and δk^*. This is illustrated below.

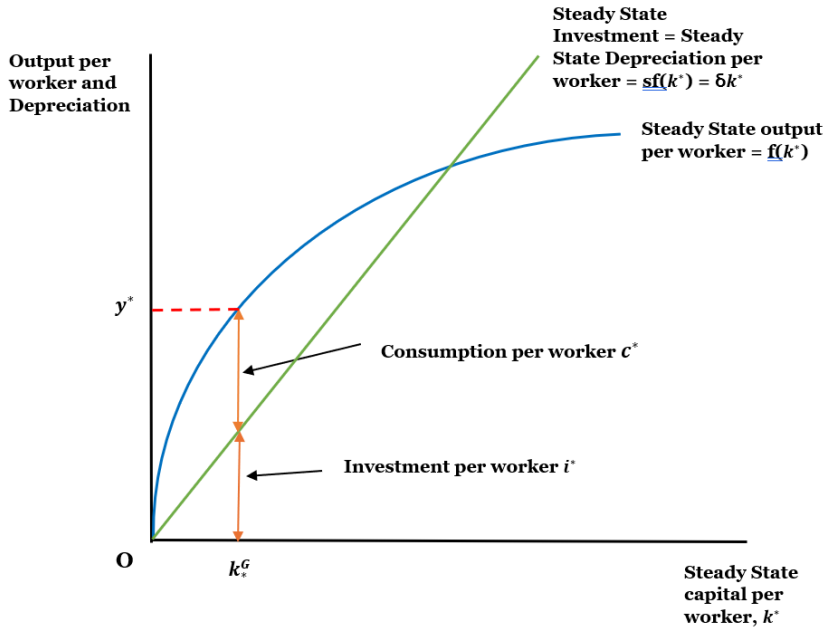


Figure 6. Golden Rule Consumption Maximising Capital Per Worker

In this diagram, k_*^G is the Golden Rule steady state level of capital per worker. At this level of capital per worker the vertical distance between the depreciation = investment line and the output per worker line is maximised. Output per worker is y^* and consumption per worker is c^*. At this point the gradient of the output per worker line and the investment/depreciation line is equal. If the steady state level of capital per worker were *below* k_*^G then an increase in the savings rate and hence capital per worker would increase output per person more than it would increase depreciation per person and so the amount available for consumption per person would increase. Society would be better off if the capital/labour ratio increased towards k_*^G And if the steady state capital per worker rose *above* k_*^G then depreciation would rise by more than the increase in output per worker and so the amount left for consumption per worker would fall and people would be less well off in welfare terms than at k_*^G. Consumption per person is maximised at k_*^G and this is accordingly the socially optimal steady state level of capital per worker. And the investment and hence saving rate corresponding to this is thus the optimal saving rate for the society.

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At the Golden Rule level of capital per worker, the gradient of the production function and depreciation line are equal. The gradient of the output curve at k_*^G is $\mathrm{d}y/\mathrm{d}k$ which is the Marginal Product of Capital (MPK), while the gradient of the depreciation=investment line is δ. Hence, at k_*^G

$$\mathrm{MPK} = \delta \qquad (25)$$

which means that, at the Golden Rule level of capital per worker, the Marginal Product of Capital is equal to the depreciation rate. If MPK > δ then an increase in capital per worker would increase output by more than the addition to depreciation and this extra output can be added to consumption:

$$\Delta \mathrm{c} = \mathrm{MPK} - \delta \qquad (26)$$

This means the economy is *not* at its consumption maximising steady state. By contrast, if MPK < δ, then an addition to capital per worker will add more to depreciation than it will to output, and the amount available for consumption will decline. Only when MPK = δ is the economy at a steady-state level of capital per worker which maximises consumption.

However, there is nothing to ensure that an economy will be at its Golden Rule steady-state level of capital per worker. Remember, all points along the depreciation line δk^* represent steady states where investment = depreciation. At the Golden Rule level of capital per worker k_*^G the level of investment (and saving) per worker is i^*. If investment (and saving) per worker were higher than i^* then steady state capital worker would be higher than k_*^G and consumption per worker would be lower. If investment were lower than i^* then consumption per worker would also be lower. In either case, the Golden Rule level of capital per worker would not be attained and there is nothing in a free-market economy to ensure that investment per worker *is* i^*.

Calculating the Golden Rule Steady State Saving Rate

We can calculate the saving rate for achieving k_*^G as follows.

Let us assume that production can be described with a Cobb-Douglas production function with constant returns and no technical progress:

$$\mathrm{Y} = \mathrm{K}^{0.5}\mathrm{L}^{0.5} \qquad (19)$$

We have already seen that in this case output per worker is a function of capital per worker:

$$y = k^{0.5}$$

$$y = \sqrt{k}$$

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Output per worker is equal to the square root of the capital per worker. This is our production function. Now, we know that the Golden Rule steady state capital per worker exists when:

$\mathrm{MPK} = \delta$

Let us assume that the rate of depreciation is 10% per annum or $\delta = 0.1$. Hence, in the Golden Rule steady state:

$\mathrm{MPK} = 0.1$

Given that $y = k^{0.5}$ then to find the marginal product of capital we differentiate y by k to get:

$$\frac{\mathrm{dy}}{\mathrm{dk}} = 0.5\mathrm{k}^{-0.5} = \frac{1}{2}\mathrm{k}^{-\frac{1}{2}} = \frac{1}{2\mathrm{k}^{\frac{1}{2}}}$$

In which case,

$\mathrm{MPK} = 0.1$

$$\frac{1}{2\mathrm{k}^{\frac{1}{2}}} = 0.1$$

Squaring throughout gives us:

$$\frac{1}{4k} = 0.01$$

$1 = 0.04k$

$k = 25$

Thus, the steady state capital per worker that yields the Golden Rule of consumption maximisation is 25:

$k_*^G = 25$

The question then is: what rate of saving yields a capital per worker of 25 given the depreciation rate of 0.1?

From the production function, we know that:

$y = \sqrt{k}$

So with a steady state capital per worker of 25, then:

$y^* = \sqrt{25} = 5$

The Golden Rule steady state output per worker is 5.

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Now the steady state exists when investment per worker = depreciation per worker, that is:

$i^* = \delta k^*$

This states that steady state investment per worker is equal to steady state depreciation per worker. Investment per worker is equal to the level of saving per worker at the steady state output per worker, which is sy^*. We can therefore write:

$sy^* = \delta k^*$

We know that $\delta = 0.1$ and $k^* = 25$, so:

$sy^* = 0.1(25) = 2.5$

And we also know that:

$y^* = 5$

Hence:

$s(5) = 2.5$

$s = 0.5$

Thus, given the steady state capital per worker of 25, and a rate of depreciation of 0.1, and the production function $y = \sqrt{k}$, the rate of saving required to yield the Golden Rule capital per worker k_*^G is 0.5 or 50 per cent. Only if the saving rate out or income is 0.5 will investment be at the level required to maximise consumption given a steady state where investment = depreciation.

Population Growth

Despite being a model of economic growth, the basic Solow model does not yield sustained economic growth. Yes, if investment exceeds depreciation, capital per worker and output per worker will rise until the steady-state capital per worker (where investment = depreciation) is reached. Also, if saving increases then capital-per worker will rise until a new higher steady-state level of capital per worker is reached. But, in each case, growth is temporary and occurs only as the economy moves *towards* equilibrium. Once it has arrived at equilibrium, where $i = \delta k$, then steady-state levels of capital per person are attained and per capita growth ceases. These are essentially comparative static results, not continuous growth processes.

To yield economic growth as a continuous process we need to expand the model – first by including population growth.

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The effect of population growth is to *reduce* capital per worker k. Imagine the population growth rate (n) is 1 per cent per annum, so that n = 0.01. Since population is rising, there are now *two* processes acting to reduce capital per worker – depreciation and a rising number of workers. In this case the change in capital per worker (Δk) depends on investment per worker, depreciation, and population growth:

$$\Delta k = i - \delta k - nk$$

$$\Delta k = i - (\delta + n)k \qquad (27)$$

Investment per worker (i) increases capital per worker, while depreciation and population growth both reduce it. If the capital stock per worker is to remain constant at 'steady state' level, then investment must equal $(\delta + n)k$:

$$i = (\delta + n)k \qquad (28)$$

The actual steady state level of capital per worker k^* is thus determined by the point where the actual level of investment per worker equals the amount of investment necessary to off-set the effects of depreciation and population growth. This is illustrated in **Figure 7**.

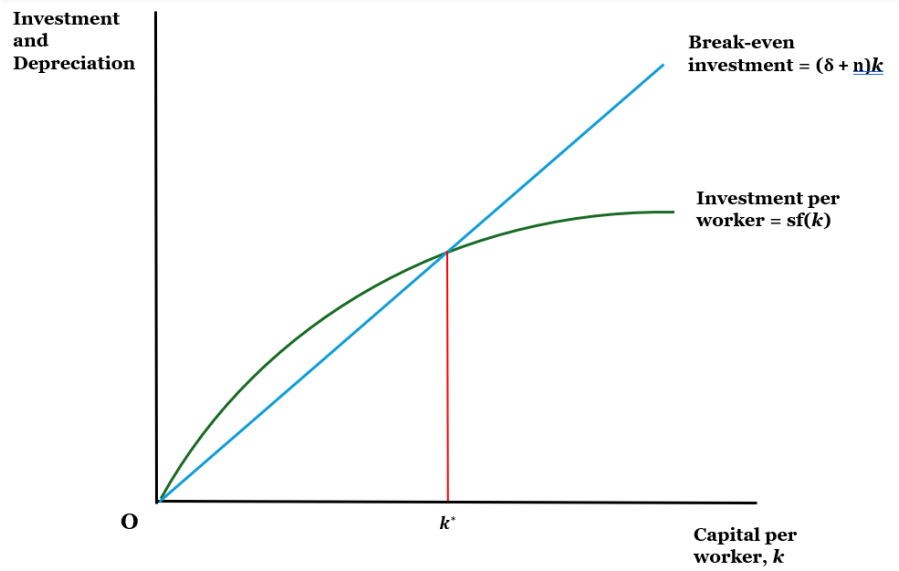


Figure 7. Steady-State Capital per worker with Depreciation and Population Growth

The $(\delta + n)k$ line is an amended version of the previous depreciation line δk. It shows the amount of investment required at any given level of capital per worker if the amount of capital per worker is to remain constant. All points along this line are potential steady-state levels of capital per worker. Previously steady state investment had to offset the effect of depreciation alone; now it must also offset the effect of population growth. It is drawn for a given level of δ and n. The equilibrium steady state level of capital per worker k^*depends on the actual level of investment which in turn depends on the saving rate s and the income per worker y since $i = sy = sf(k)$. If investment

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per worker exceeds the effects of depreciation and population growth in reducing capital per worker, then there will be a net increase in capital per worker:

$\Delta k = \text{sf}(k) - (\delta + n)k$

In the diagram, this is true at points to the left of k^* where i exceeds $(\delta + n)k$ and the capital/labour ratio rises towards k^*. By contrast, to the right of k^*investment is less than the amount needed to offset depreciation and population growth and capital per worker declines towards k^*. Only at k^* is investment equal to depreciation and population growth causing the capital/labour ratio to remain unchanged.

$\text{sf}(k^*) = (\delta + n)k^*$ (29)

Hence, $\Delta k = 0$ and this is the steady state level of capital per worker.

As Mankiw observes, the introduction of population growth has three effects on the Solow model.[1]

1. It generates sustained economic growth for the first time. In the steady state with population growth, capital per worker (k) and output per worker (y) are constant. But as the number of workers is growing at rate n then total capital net of depreciation must also be growing at rate n, and since output per worker is constant, then as the number of workers grows at rate n then total output (Y) will also grow at rate n. Hence, total output or GDP will grow at the population growth rate n. The absolute economy is getting larger – though output per worker (per capita GDP) remains constant.
2. It shows how an increased rate of population growth, other things remaining equal, causes output per worker and hence per capita GDP to decline. For a given rate of saving and investment, if the rate of population growth n increases then the steady state level of capital per worker (k^*) will decline, and as capital per worker declines so does the steady state level of output per worker y^* decline since y^*= f(k^*). This is illustrated below.

[1] Mankiw, *Macroeconomics*, pp. 233-234.

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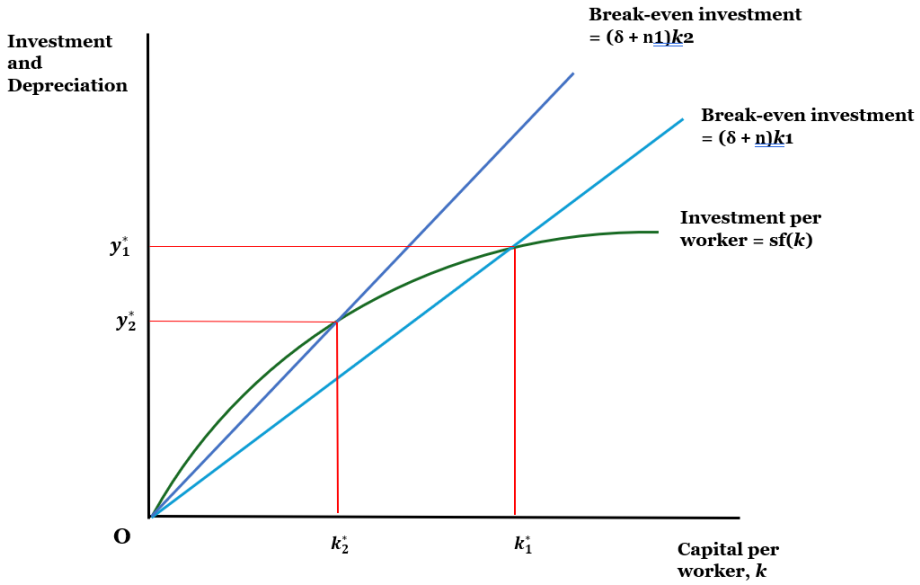


Figure 8. How Increased Rate of Population Growth Lowers Steady-State Capital per Worker

In this diagram, the economy is initially in steady-state equilibrium at capital per worker of k_1^*. The population growth rate is n and with a saving and investment function $sf(k)$ the level of capital per worker will be constant at k_1^* to which there corresponds a level of output per worker y_1^*. Suppose the population growth rate accelerates to n_1. The level of investment per worker is no longer sufficient to offset the effects of depreciation and population growth, so capital per worker contracts. As capital per worker contracts depreciation declines and eventually at k_2^* the level of investment per worker is just sufficient to maintain the capital-labour ratio at a constant level. This is the new steady-state level of capital per worker. Thus, the effect of accelerated population growth is to reduce capital per worker and reduce output per worker. A country with faster population growth will, other things being equal, tend to have a lower per capita GDP. On the other hand, if the population growth rate decelerated from n_1 to n then investment per worker would exceed break-even levels and capital per worker would rise to k_1^*, raising per capita GDP. This would be an effect, for example, of China's one-child policy when first introduced – causing an upward shift in per capita GDP.

3. Population growth affects the consumption maximising Golden Rule level of capital per worker. Remember that:

$$c^* = y^* - i^*$$

Which is to say, steady state consumption per worker equals steady state output per worker minus steady state investment per worker. Given that:

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$i^* = (\delta + n)k^*$

$y^* = f(k^*)$

then:

$c^* = f(k^*) - (\delta + n)k^*$ (30)

We saw before that the Golden Rule level of capital per worker existed when:

$MPK = \delta$ (25)

Now that we have added the effect of population growth to depreciation, the Golden Rule steady state level of capital worker k_G^* exists when:

$MPK = \delta + n$ (31)

$MPK - \delta = n$

Thus, for the Golden Rule consumption-maximising level of capital per worker, it is necessary that the marginal product of capital per worker net of depreciation is equal to the rate of population growth. For a given rate of depreciation δ, the higher is the rate of population growth the higher must be the MPK, which means that the Golden Rule capital per worker (k_G^*) must be lower, since diminishing returns to capital means that for the MPK to increase the capital/labour ratio must fall.

Technological Progress

So far we have looked at the effect of capital accumulation and population growth on output per worker. We now introduce a third source of change in output per worker: namely, technical change. Solow models the effect of technical progress through its effect in enhancing the productivity of labour. Technical change means a given labour input adds more to output than before. Hence, it is said to be 'labour augmenting'.

Previously our production function for total output was:

$Y = F(K, L)$

We now amend this by adjusting the labour input according to the efficiency of Labour, A, where A includes society's knowledge about production methods, including the use of new machines. Thus:

$Y = F(K, AL)$

The efficiency factor (A) captures the productivity of Labour due to improved methods of production, use of more advanced machinery, better education, and improved health of workers. As the efficiency of labour rises each hour of work contributes more to production and by multiplying L and A we arrive at a measure of *Effective Labour*

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inputs, capturing not only the number of workers but the effectiveness of their work. So, total output now depends on the quantity of capital (K) and the effective labour input (AL). An increase in the efficiency of labour is equivalent to an increase in the number of workers. There are two ways of looking at the effect of this:

1. If we hold total output Y constant, then an increase in the effectiveness of Labour due to technical change (A) will *reduce* the quantity of Labour input required to make Y. If A doubles then the productivity of Labour will double and only half the original number of workers will be required to produce a given output.
2. If we increase the technical efficiency of Labour while holding the number of workers constant then total output Y will increase.

Technical progress that increases the efficiency of Labour is called *labour augmenting* technical progress. If the efficiency of labour increases by 2 per cent per annum, then we can say that the efficiency of labour grows by a constant rate g = 0.02, which means that each unit of labour is 2 per cent more productive each year and output grows as if the labour force is growing by 2 per cent a year. Hence, if the labour force is growing by *n* each year and the efficiency of labour is growing by g, then the effective labour force is growing at:

$$\dot{\mathrm{AL}} = n + g \qquad (32)$$

Here $\dot{\mathrm{AL}}$ is the rate of growth of the effective labour force. Introducing the concept of *effective labour* means that we can think not of output per worker but output per effective worker, and capital per effective worker and so on. Thus:

$\frac{\mathrm{Y}}{\mathrm{L}}$ becomes $\frac{\mathrm{Y}}{\mathrm{AL}}$

And since:

$\mathrm{Y} = \mathrm{F}(\mathrm{K}, \mathrm{AL})$

then:

$$\frac{\mathrm{Y}}{\mathrm{AL}} = \mathrm{F}\left(\frac{\mathrm{K}}{\mathrm{AL}}, 1\right) = \mathrm{F}\left(\frac{\mathrm{K}}{\mathrm{AL}}\right)$$

Writing y_A for Y/AL and k_A for K/AL, we can write:

$$y_A = \mathrm{f}(k_A) \qquad (33)$$

This means that output per effective worker (y_A) is a function of capital per effective worker (k_A).

In terms of our Cobb-Douglas production function where:

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$$Y = K^{0.5}L^{0.5}$$

This now becomes:

$$Y = K^{0.5}AL^{0.5} \qquad (34)$$

Dividing through by AL we get:

$$\frac{Y}{AL} = \frac{K^{0.5}AL^{0.5}}{AL} = K^{0.5}AL^{-0.5} = \frac{K^{0.5}}{AL^{0.5}} = \left(\frac{K}{AL}\right)^{0.5}$$

Since Y/AL can be written as y_A and K/AL is k_A, then:

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.5}$$

Becomes:

$$y_A = (k_A)^{0.5} \qquad (35)$$

Thus, output per effective worker is the square root of capital per effective worker.

Our production function relating output per effective worker to capital per effective worker can be drawn as follows.

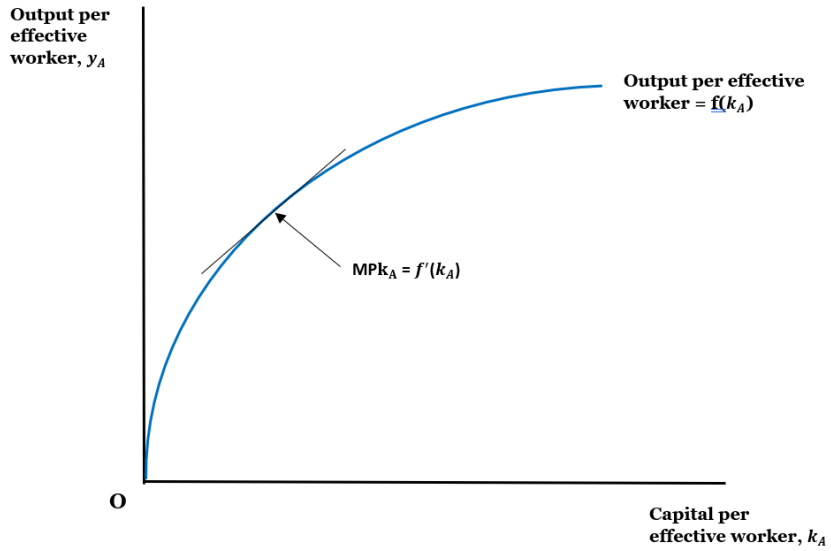


Figure 9. Output per Effective Worker as a Function of Capital per Effective Worker

As before, as capital per effective worker increases total output per worker rises, but at a declining rate due to diminishing returns.

Technical progress causing a rise in the quantity of effective labour (AL) means that capital per effective worker is subject to three diluting effects: depreciation of capital, increase in number of workers, and increase in effective workers. Previously it was depreciation of capital and a rising population that reduced capital per worker. But since we are now dealing with effective labour inputs, a rise in effective labour inputs

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due to more productive technology will also act to reduce capital per effective labour inputs. This means that the break-even quantity of investment per effective worker will need to be higher now that technology is no longer constant but increasing. In other words:

$$\Delta k_A = \text{sf}(k_A) - (\delta + n + g)k_A \qquad (36)$$

David Romer describes this as the 'key equation of the Solow model.'[1] Break-even investment per effective worker is $(\delta + n + g)k_A$. If capital per effective worker k_A is to increase, investment per worker must exceed the dilution effects of depreciation (δ), population growth (n), and the rate of labour-augmenting technical progress (g). The steady-state level of capital per effective worker k_A^* exists when investment per effective worker equals the break-even investment per worker:

$$\text{sf}(k_A) = (\delta + n + g)k_A \qquad (37)$$

Again, we can depict this in diagrammatic form:

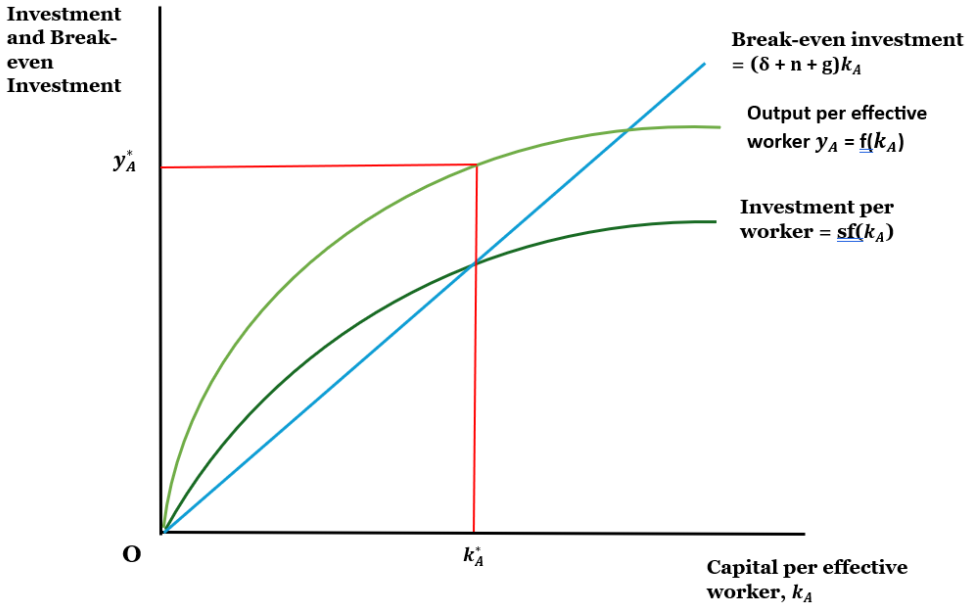


Figure 10. Steady-State Capital per Effective Worker

Depreciation, population growth, and technical progress all tend to reduce the amount of capital per effective worker. To offset this investment is required. When investment = $\text{sf}(k_A) = (\delta + n + g)\ k_A$ then capital per effective worker is stabilised and a steady-state level of capital per effective worker, k_A^*, is attained. k_A^* is the level of capital per effective worker where the ratio of capital to effective worker is stabilised as is output per effective worker:

[1] D. Romer, *Advanced Macroeconomics*, p. 15.

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$y_A^* = \mathrm{f}(k_A^*)$

An economy is either *at* the steady-state level of capital per effective worker or tending towards it. If capital per effective worker is below k_A^* then investment per effective worker exceeds the reducing effects of depreciation, population growth, and technical change, and so capital per effective worker will rise. And if capital per effective worker is above k_A^* then investment per effective worker is below break-even levels and capital per effective worker will contract towards k_A^*. Since capital per effective worker is constant at the steady state, and since effective labour inputs are growing at $n + g$, then the capital stock K must be growing at $n + g$. And since total capital (K) and effective labour in inputs (AL) are both growing at $n + g$, then given constant returns, Y or total output is also growing at $n + g$. The economy is on a *balanced growth path*, with output, capital, and effective labour all growing at the rate $n + g$.[1]
To revert to *actual* workers, we know that output per worker is:

$\frac{\mathrm{Y}}{\mathrm{L}} = y$

And output per effective worker is:

$\frac{\mathrm{Y}}{\mathrm{AL}} = y_A$

In which case:

$\frac{\mathrm{Y}}{\mathrm{L}} = \mathrm{A}y_A \qquad (38)$

Which is output per actual worker. So we can now say that, since output per effective worker (y_A) is constant at the steady state, then output per worker or labour productivity (Y/L) grows at the same rate as A, which is the growth rate of the efficiency of labour. Hence, the effect of introducing technical change into the Solow model is to yield a steady-state growth rate per worker equal to the rate of technical change. If we combine this result with an increase in L due to population growth, we get an expression for the growth of absolute GDP as follows:

$\frac{\mathrm{Y}}{\mathrm{L}} = \mathrm{A}y_A$

$\mathrm{Y} = \mathrm{LA}y_A \qquad (39)$

Output per effective worker (y_A) is constant in the steady state, but A grows by g and L grows by n, so Y grows as the labour force L grows and technical progress occurs; i.e.

$\dot{\mathrm{Y}} = g + n$

[1] *Ibid*., pp. 16-17.

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Thus, in the Solow model, the rate of growth of GDP is explained by growth in population and technical progress, the two together representing the increase in effective labour inputs. Subtracting the rate of population growth (n) we get:

$$\dot{Y} - n = g$$

This means that the rate of growth of total output exceeds the rate of growth of population and labour supply n by the amount g, which is accordingly the growth of output per worker. Thus, it is technical progress that yields rising GDP per person and thus rising living standards.

In terms of the Golden Rule level of capital per effective worker which maximises consumption per person, we now have:

$$c^* = f(k_A^*) - (\delta + n + g)k_A^* \qquad (40)$$

Steady state consumption is maximised when:

$$MPK = \delta + n + g \qquad (41)$$

Or

$$MPK - \delta = n + g$$

This states that for the Golden Rule level of capital per effective worker, the marginal product of capital per effective worker after deducting for depreciation must equal the sum of the rate of population growth and technical change.

Steady-State and Golden Rule Rates with a Cobb-Douglas Production Function

We saw above that the constant returns labour-augmenting Cobb-Douglas production function can be written as:

$$Y = K^{0.5}AL^{0.5} \qquad (34)$$

Expressing this in terms of output per effective worker, we got:

$$y_A = (k_A)^{0.5} \qquad (35)$$

Or, with steady state values:

$$y_A^* = (k_A^*)^{0.5}$$

Steady-state capital per effective worker exists when investment per effective worker equals depreciation, population growth and technical change per effective worker:

$$s(k_A^*)^{0.5} = (\delta + n + g)k_A^*$$

Rearranging this expression to determine for k_A^*

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$$\frac{s}{\delta + n + g}(k_A^*)^{0.5} = k_A^*$$

$$\left(\frac{s}{\delta + n + g}\right)^2 k_A^* = (k_A^*)^2$$

$$\left(\frac{s}{\delta + n + g}\right)^2 = \frac{(k_A^*)^2}{k_A^*}$$

$$k_A^* = \left(\frac{s}{\delta + n + g}\right)^2 \qquad (42)$$

Thus, an increase in the saving rate (s) will increase the steady-state level of capital per effective worker, while increases in the rate of depreciation (δ), population growth (n) and technical progress (g) all tend to reduce it.

Let us now calculate the steady state Golden Rule level of capital per effective worker. The Golden Rule condition is:

$$\text{MPK} = \delta + n + g \qquad (41)$$

Where MPK is the marginal product per effective worker due to a change in capital per effective worker; i.e.

$$\text{MPK} = \frac{dy_A}{dk_A}$$

This, we have seen, is equal to:

$$\frac{d}{dk_A}(k_A)^{0.5} = \ 0.5(k_A)^{-0.5} = \frac{1}{2k_A^{0.5}}$$

We continue to assume that the rate of depreciation is 10% per annum or δ = 0.1. Now also assume that the rate of population growth (n) is 1% per annum and technical progress (A) is 3% per annum. Thus our Golden Rule condition becomes:

$$\text{MPK} = 0.1 + 0.01 + 0.03$$

$$\text{MPK} = 0.14$$

Hence, the Golden Rule level of capital per effective worker is that level where:

$$\frac{1}{2k_A^{0.5}} = 0.14$$

Squaring throughout this becomes:

$$\frac{1}{4k_A} = 0.02$$

In which case:

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$1 = 0.02(4k_A)$

$1 = 0.08k_A$

$k_A^* = 12.5$

Thus, the steady-state level of capital per effective worker that maximises consumption per worker is 12.5. What saving rate yields a capital per effective worker of 12.5 given the dilution rate of 0.14? To calculate this we first determine the steady state output per effective worker given k_A^* = 12.5.

$y_A^* = (k_A^*)^{0.5}$

$y_A^* = (12.5)^{0.5}$

$y_A^* = 3.5$

This is the steady state Golden Rule output per effective worker.

A steady state exists when investment per effective worker just offsets depreciation per worker, population growth, and technical change:

$i_A^* = (\delta + n + g)k_A^*$ (35)

Investment per effective worker is equal to the level of saving per effective worker at the steady state output per effective worker, which is sy_A^*. We can therefore write:

$sy_A^* = (\delta + n + g)k_A^*$

We know that $\delta = 0.1$, $n = 0.01$, and $g = 0.03$, and $k_A^* = 12.5$. So:

$sy_A^* = 0.14(12.5) = 1.75$

Given that:

$y_A^* = 3.5$

then:

$s(3.5) = 1.75$

$s = 0.5$

Thus, a saving rate of 50 per cent will be required to ensure the Golden Rule level of consumption per person maximising steady state capital per effective worker.

Interestingly, the effect of adding population growth and technical change to the Solow model is to cause the Golden Rule level of capital per effective worker to decline. This is because, in going from MPK = δ to MPK = $\delta + n + g$ the right-hand side of the expression increases, and so MPK must increase, and due to diminishing returns to

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capital per effective worker, to raise MPK then capital per effective worker must fall. However, actual output per worker will rise due to technical change. Since:

$$y_A = \frac{\mathrm{Y}}{\mathrm{AL}}$$

$$\text{then } y_A \mathrm{A} = \frac{\mathrm{Y}}{\mathrm{L}} = y$$

$\mathrm{Y/L}$ (or y) is output per actual worker, which equals output per effective worker y_A multiplied by technical progress (A). Since output per effective worker is constant at the steady state rate of growth, but A rises by g, then output per worker y will rise by g over time. In our example that is 3 per cent per annum.

Mankiw has applied the Solow model to the US economy to determine whether the US rate of saving is at the level required for the Golden Rule level of steady-state capital per worker.[1] The US economy grows at about 3 per cent per year, which means that $n + g$ = 3 per cent per annum. The annual depreciation rate he puts at 4 per cent per annum and the Marginal Product of Capital (MPK) at 12 per cent. The net result is:

$$\mathrm{MPK} - \delta = n + g$$

$$0.12 - 0.04 > 0.03$$

$$0.08 > 0.03$$

This suggests that, in the US, the MPK is too large relative to the rates of population and technological growth to yield a Golden Rule level of consumption per person. If the Golden Rule is to be attained the level of capital per worker needs to rise since this would lower the marginal product of capital. In short, the US saving rate, which in 2016 stood at 17 per cent of GDP, would have to rise if it were to arrive at a higher level of capital per effective worker and maximise its consumption per person. It is possible that the current American AI boom will help to close this gap if the saving rate rises and the rate of technical progress accelerates. In 2026 the saving rate has indeed risen to 18.7 per cent. However, the growth rate of GDP in 2026 is 2.2 per cent, lower than the trend Mankiw assumes. It thus remains likely that capital per worker in the US continues to be below Golden Rule steady state levels.

We can similarly use the Solow model to account for China's rapid economic growth since the 1990s.[2] For steady state capital per actual worker we know that:

$$sy = i = (\delta + n + g)k$$

[1] Mankiw, *Macroeconomics*, pp. 251-52.
[2] Blanchard *et. al*., *Macroeconomics*, p. 339.

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which states that saving (and hence investment) per worker equals the rate of dilution of capital per worker. Dividing both sides by *y* we have:

$$\frac{sy}{y} = (\delta + n + g)\frac{k}{y}$$

$$s = (\delta + n + g)\frac{\frac{K}{L}}{\frac{Y}{L}}$$

$$s = (\delta + n + g)\frac{K}{Y}$$

This is the balanced growth equation showing the saving rate s necessary to maintain capital per worker at a constant level. In the case of China, it has been estimated that for the period 1995-2007 the rate of depreciation was 0.05, the rate of labour supply growth was 0.005, and the rate of technical progress was 0.06. The ratio of capital stock to total GDP was 2.6. Hence we arrive at:

$$s = (0.05 + 0.005 + 0.06)(2.6)$$

$$s = (0.115)(2.6)$$

$$s = 0.3$$

Thus, to maintain steady state growth with a constant capital/labour ratio, China needed to save 30 per cent of its income. In fact, China saved 39 per cent of its income. This means that China's capital accumulation rate exceeds steady-state levels, resulting in rising capital per worker. It is this, combined with a rapid technical progress rate of 6 per cent (over double the US figure) that accounts for China's rapid growth rates (10 per cent per annum in the period 1995-2007).[1] Yet, despite the high rate of investment per worker, China's capital/labour ratio is still below the Golden Rule level which will maximise consumption per person. For the Golden Rule level of capital per worker it is necessary that:

$$MPK - \delta = n + g$$

In the case of China in 2005, the IMF calculated the rate of return on capital to be 20 per cent, so the MPK = 0.20.[2] Thus:

$$0.20 - 0.05 > 0.005 + 0.06$$

$$0.15 > 0.065$$

Thus, as in the case of the US, the MPK minus the rate of depreciation exceeds the rate of labour supply growth and technical change by a factor of 2.3. This suggests

[1] Blanchard et. al., *Macroeconomics*, p. 339.

[2] A. Singh, M. Naber, and P. N'Diaye, *China's Economy in Transition* (2013), accessed at China's Economy in Transition – From External to Internal Rebalancing | IMF eLibrary.

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that China needs to further increase capital per worker further to reduce the MPK and increase consumption per person. China's high investment rate *has* reduced the MPK in China – which was estimated at 0.25 for the period 1973-93. That China's growth rate should exceed that of mature economies like the USA and Europe is something the Solow model predicts. Referring to **Figure 10**, when an economy's capital per effective worker is below the steady-state level k_* then its capital per worker will rise towards the steady state level and during this period its growth rate will exceed that of developed economies already at the steady state. But the model also predicts that over time China's growth rate will converge towards that of other developed economies. As capital per worker rises the MPK of capital will decline (as China's has), while once China has caught up in deploying the most advanced technology, its rate of technical progress will equal the *g* of developed economies. The Solow model thus predicts that over time all economies will converge on the same balanced growth path with the same levels of technology and growth of GDP equal to *n* + *g*.

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